Section 2.2 #9b (modified denominator to illustrate 17a)

Can the function

$$f(x,y) = \frac{\cos(xy)-1}{xy}$$

be made continuous everywhere by suitably defining it when $xy = 0$?

Solution: When $xy = 0$ we know that either $x = 0$ or $y = 0$, hence we will have problems defining this function along the y and x axis. However, it pays to notice that $f : \mathbb{R}^2 \to \mathbb{R}$ can be written as the composition $f = g \circ h$ where $h : \mathbb{R}^2 \to \mathbb{R}$ is defined by $h(x, y) = xy$ and $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(\alpha) = \frac{\cos(\alpha)-1}{\alpha}$.

By Theorem 5 in section 2.2 we know that if h is continuous at a point (x_0, y_0) and g is continuous at the image point $h(x_0, y_0)$ then the composition $g \circ h$ will be continuous at the point (x_0, y_0). Hence we only need to show that g and h are continuous at every point in their domains in order to show that f is continuous at every point (or, more simply, f is continuous).

The fact that h is continuous everywhere is easy - it is shown in Example 8 on page 119.

The fact that g is continuous everywhere is only slightly more difficult. Obviously g has a problem when $\alpha = 0$. However, if we use L'Hopital's rule then we see that g can be MADE continous by redefining it as

$$g = \left\{ \begin{array}{cc} \frac{\cos(\alpha)-1}{\alpha} & \alpha \neq 0 \\ 0 & \alpha = 0 \end{array} \right\}.$$

Hence we have shown that g and h are continuous everywhere on their respective domains, hence $f = g \circ h$ is continuous everywhere.