CLASS NOTES PART I (M427K FALL 2004)

Introduction

Due to popular request, I have decided to post an outline/summary of what we have done so far in class. I will continue to update this resource as the class continues. I hope that this will more or less "tie" the notes and homeworks together into something cohesive - something that you can understand.

Essentially, this will be zoology. I will classify what we have done so far, and put in references to relevant homework problems. There has been a LOT of material covered in class but not in the homeworks. Here I will try to outline briefly what to do if you come across this stuff, and hopefully put in an example.

I know that this class is demanding. Just do your best... and keep in mind that I'm spending more time typing this stuff up than you are in learning it (and that I have my own work too).

1. Variable Separable Type

These type of equations are called Variable Separable because we can (more or less) easily separate the variables and just integrate. At the time of this writing the class is already well beyond this, so I won't spend much time laboring over the details.

1.1. **Variable Separable.**

These are the easiest. By using basic high school algebra we can separate the equation

$$M(x,y)dx + N(x,y)dy = 0$$

(where $M(x,y)$ and $N(x,y)$ both depend on x *and* y) into something that looks like

$$M'(x)dx + N'(y)dy = 0.$$

This should be easy to integrate and solve. For examples see Homework #1 problems 1, 2, and 3.

link `www.math.utexas.edu/~stirling/teaching/M427K/hw1.pdf`

1.2. **Homogeneous Equation.**

This is the "kissing cousin" of type 1.1. It is of the form

$$M(x,y)dx + N(x,y)dy = 0$$

where $M(x,y)$ and $N(x,y)$ are *homogeneous* (of the SAME order). If you don't remember what homogeneous means, then consult YOUR notes. This is only meant to be a brief outline.

The strategy to solve these is to try the substitution $y = ux$. For an example of this stuff see Homework #2 problem 3.

link `www.math.utexas.edu/~stirling/teaching/M427K/hw2.pdf`

1.3. **"Almost" Homogeneous Equation.** This is "almost" like type 1.2. It looks like

$$(ax + by + 5)dx + (cx + dy + 7)dy = 0$$

(notice that the additional constants 5 and 7 make the whole equation non-homogeneous). The trick is to change variables so that it looks like type 1.2 (remember the crossing lines), and then solve it using the method presented there. For examples see Homework #3 and Homework #4 problems 1 and 2.

link `www.math.utexas.edu/~stirling/hw3sols.pdf`

link `www.math.utexas.edu/~stirling/teaching/M427K/hw4sols.pdf`

NOW there is one possible problem... that is IF the lines don't cross (if the lines are parallel). Then, using this method, you will find that you won't be able to change coordinates to make it homogeneous (you will end up with something ridiculous like 2=0... something that isn't true). No homeworks were presented with this sort of problem, so let me just give an example

1.3.1. *Example of when lines don't cross.*

$$(x + y + 3)dx + (2x + 2y + 7)dy = 0$$

This *looks* "almost" homogeneous, so let's try the substitution $x = \overline{x} + h$ and $y = \overline{y} + k$ like in Homework #3. You will try to solve for h and k and you will just get the equation $3 = \frac{7}{2}$. This is bogus... so this method didn't work (the lines are parallel).

Fortunately, the answer will be even easier... try the substitution $v = x + y + 3$ (or whatever comes in front of the "dx") in the differential equation. So $dv = dx + dy$ (hence $dx = dy - dv$). Now get rid of x in favor of v in the differential equation and you will see that it will become completely separable (type 1.1).

2. Exact Differential Equation Type

We sort of breezed over these, but they're going to show up on the exams, so listen up. Basically we need to classify these by how we solve them

2.1. **Exact Differential Equation.** These are equations of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where the "exactness test" is true, that is: $\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$. I wrote a long discussion of these equations (and how to solve them) in Homework #4 problem 3

link `www.math.utexas.edu/~stirling/teaching/M427K/hw4sols.pdf`

2.2. **Integrating Factor Method.** These are differential equations of the form

$$y'(x) + P(x)y(x) = Q(x).$$

They can be solved by using the so-called "integrating factor function" $\mu(x)$. For details on how to solve these see Homework #6 problems 1, 2, and 3.

link `www.math.utexas.edu/~stirling/teaching/M427K/hw6sols.pdf`

2.2.1. *Bernoulli Equation.* This is really just making type 2.2 a little fancy. There were no homework problems about it, so here's what to do: if you see a differential equation of the form

$$y'(x) + P(x)y(x) = Q(x)y^n(x)$$

then notice how much it looks like type 2.2 (it just has an extra $y^n(x)$ on the RHS).

Try doing the obvious thing: divide by $y^n(x)$...

$$\frac{y'(x)}{y^n(x)} + P(x)\frac{y(x)}{y^n(x)} = Q(x).$$

Then let $Y = \frac{y(x)}{y^n(x)} = y^{1-n}(x)$. So then $Y'(x) \equiv \frac{dY}{dx} = \frac{dY}{dy}\frac{dy}{dx} = (1-n)y^{(1-n)-1}(x)y'(x) = (1-n)\frac{y'(x)}{y^n(x)}$. Ah ha!! In other words, $\frac{y'(x)}{y^n(x)} = \frac{Y'(x)}{(1-n)}$, so let's just plug these results into the differential equation, yielding:

$$\frac{Y'(x)}{(1-n)} + P(x)Y(x) = Q(x).$$

Except for the factor of $(1-n)$ (which is just a number, after all, so we can multiply the whole equation by it giving a *new* $P(x)$ and $Q(x)$) this is EXACTLY type 2.2. So NOW solve it using the method given there.

3. Nth Order Linear DiffEQ

So far you have been studying how to solve *first order (1 derivative) differential equations.* These methods can be used to solve BOTH linear AND nonlinear equations (but only if there is at most 1 derivative!!!). If you don't know what the words "linear" and "nonlinear" mean (which I suspect that you DON'T), don't freak out. Basically, linear equations are MUCH easier to solve than nonlinear equations.

Now, if we allow higher derivatives ("nth order differential equations") then it becomes MUCH harder. We actually can't take care of nonlinear equations anymore, but fortunately we can still handle linear equations. This will be the subject of most of the rest of the class (I suspect).

We'll start by solving the "n^{th} order linear differential equations" by first considering the ones when the RHS=0. Then we'll try to take care of them when the RHS is NOT zero.

3.1. n^{th} Order Linear Differential Equations with RHS=0.

We'll start by limiting ourselves to only 2 derivatives (and with the RHS=0). These are equations that look like

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0,$$

or we usually write it in the shorthand

$$y'' + 5y' + 6y = 0.$$

The RHS=0 condition just means that, after we put all of the y's onto the LHS, there is nothing left on the RHS. Don't be fooled by some equation like $y'' + 5y' = -6y$ (which is just the same equation... you might not think that the RHS is zero, but just take the y stuff over to the other side).

Now sometimes we also write the derivative operator $\frac{d}{dx}$ as a big "D". So we could rewrite this equation as

$$(D^2 + 5D + 6)y(x) = 0.$$

However you want to write it, there are a couple of ways to solve this. You could factor the derivative operators, yielding

$$(D+2)[(D+3)y(x)] = 0$$

and then consider it like peeling an onion (starting from the outside, and moving in). I mean that you could solve first $(D+2)v(x) = 0$ for $v(x)$ and then plug this solution into the equation $[(D+3)y(x)] = v(x)$ to solve for $y(x)$. In fact, this is what we'll need to do later.

3.1.1. *Solve by guessing.* The easier way is to just guess a solution: try $y(x) = e^{mx}$ and plug it into the differential equation to figure out what m has to be. This gives the equation

$$m^2 e^{mx} + 5me^{mx} + 6e^{mx} = 0.$$

Now factor out the e^{mx}, giving

$$e^{mx}(m^2 + 5m + 6) = 0.$$

So $m^2 + 5m + 6 = 0$, which has solutions (quadratic formula) $m = -2, -3$. So I have solutions to the differential equation:

$$y(x) = e^{-2x}, \;\; y(x) = e^{-3x}.$$

In the math lingo, these are "linearly independent solutions", so the real solution is just these added together with arbitrary constants out front:

$$y(x) = C_1 e^{-2x} + C_2 e^{-3x}.$$

3.1.2. *What if we have same roots?* Now it's not always true that we will have different roots, i.e. we might have had $m = -2, -2$. The differential equation that would give these two (same) roots is

$$(D^2 + 4D + 4)y(x) = 0.$$

This is a bit more messy, and we need to use what we briefly mentioned (before we played the guessing game $y(x) = e^{mx}$). We need to solve this by factoring the operator, i.e. making it look like $(D+2)[(D+2)y(x)] = 0$. Then we solve these differential equations a factor at a time (starting from the outside). So we start by letting $v(x) = [(D+2)y(x)]$ (the stuff inside the [] brackets). Then we have the equation $(D+2)v(x) = 0$. This is really easy to solve... it is first order (it has only 1 derivative), so it falls into the category of what we've been doing all semester already. In fact, it's separable, with the solution

$$v(x) = C_1 e^{-2x}.$$

Plugging this into the equation $v(x) = [(D+2)y(x)]$ gives the equation

$$C_1 e^{-2x} = (D+2)y(x).$$

This also is a first order differential equation, so you should have enough tools to solve it (hint: use method of integrating factors). The full solution is

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x}.$$

Notice the extra x on the second term. This is a trend: for every repeated root we need to multiply by x, x^2, x^3, x^4, ...

So if we had the 4th order differential equation

$$(D+2)(D+3)(D+2)(D+2)y(x)$$

then we would have $m = -2, -2, -2, -3$. So the general solution would be

$$y(x) = C_1 e^{-3x} + C_2 e^{-2x} + C_3 x e^{-2x} + C_4 x^2 e^{-2x}.$$

3.1.3. *What if roots are complex?* We might also have complex roots when we factor the auxilliary equation, i.e. we might have

$$m = a + ib,\ a - ib.$$

We do the *same thing...* the solution is just these roots in an exponent with arbitrary coefficients out front

$$y(x) = C_1 e^{(a+ib)x} + C_2 e^{(a-ib)x}. \tag{3.1}$$

That's it!!! Now the coefficients C_1 and C_2 should be complex numbers (since we already introduced complex numbers into the problem, we have to live with complex coefficients everywhere).

Dr. Guy spent quite a while rewriting this into just a cosine. Just keep in mind that this is ONLY a rewriting! First, start by factoring e^{ax} out front and then using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to rewrite equation 3.1 as

$$y(x) = e^a((C_1 + C_2)\cos(bx) + i(C_1 - C_2)\sin(bx)).$$

It's at this point that Dr. Guy actually PUTS IN MORE INFORMATION!!! He said that he only wants REAL solutions (i.e. he wants to OUTLAW complex numbers). So we just change the complex coefficients in front of the sine and cosine to real numbers A and B, i.e.

$$y(x) = e^a(A\cos(bx) + B\sin(bx)).$$

Now the rest is just trig identities. Factor out $\sqrt{A^2 + B^2}$ out front, giving

$$y(x) = e^a\sqrt{A^2 + B^2}(\frac{A}{\sqrt{A^2 + B^2}}\cos(bx) + \frac{B}{\sqrt{A^2 + B^2}}\sin(bx)).$$

Then we can think of a right triangle from high school. One of the legs has length A and the other has length B. According to the Pythagorean theorem, the hypotenuse has length $\sqrt{A^2 + B^2}$. So if we call the angle inside of the triangle δ, then $\frac{A}{\sqrt{A^2+B^2}} = \cos(\delta)$ and $\frac{B}{\sqrt{A^2+B^2}} = \sin(\delta)$. Hence we rewrite this the solution as

$$y(x) = e^a\sqrt{A^2 + B^2}(\cos(\delta)\cos(bx) + \sin(\delta)\sin(bx)).$$

Letting $C = \sqrt{A^2 + B^2}$ and using the trig identities that you were given in Homework #5 yields the final result

$$y(x) = e^a D\cos(bx + \delta).$$

That's it!!! Equation 3.1 has been rewritten in terms of a cosine and arbitrary (real) constants D and δ. This is somehow a prettier way to write it than equation 3.1, because we know how that looks!!!

3.2. **On to n^{th} Order Linear Differential Equations with RHS NOT ZERO!!!** This is such a big topic that I'll present it in part II of the notes.