HOMEWORK #7 (M427K FALL 2004) (REVISION 1)

Introduction

This homework comes from the textbook (Boyce). You are required to read the examples 1, 2, 3, and 4 starting on page 52.

Then, you are supposed to solve problems 1, 3, and 4 on pages 59-60

1. Page 59 #1

Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 liters of a dye solution with a concentration of 1 g/liter. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 liters/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

1.1. Solution.

This problem is asking you to measure the concentration of the dye as a function of time. So denote the *amount* of dye (in grams) in the tank as Q and the time as t. Then

$$\frac{dQ}{dt} = \text{rate}_{in} - \text{rate}_{out}.$$

Now $\text{rate}_{in} = \text{concentration}_{in} \times \text{flowrate}_{in}$. This should be obvious to you. If not, you need to learn some basic dimensional analysis. The variable $\text{concentration}_{in}$ is in units of g/liter, and is just equal to 0 g/liter (the inflowing water is fresh water, so contains no dye). The variable flowrate_{in} has units of liters/min, and in this problem is equal to 2 liters/min. So rate_{in} is just (0 g/liter)(2 liter/min)=0 g/min.

Using a similar argument, $\text{rate}_{out} = \text{concentration}_{tank} \times \text{flowrate}_{out}$ which is just (Q g/200 liter)(2 liter/min)=$Q/100$ g/min. So the differential equation is just

$$\frac{dQ}{dt} = -Q/100.$$

This should be totally trivial to solve.

The solution is just

$$Q = Ce^{-t/100}.$$

Notice the constant C. We always just write it down without questioning what the hell it IS!!! Here we will determine it by imposing some kind of *initial condition.* Notice that at time $t = 0$ this equation reads $Q(0) = Ce^0 = C \times 1 = C$. So the concentration $Q(0)$ at time zero is just the constant C. So in this differential equation we can interpret C as the initial value of Q.

Now the problem asks to find the time when the concentration is at 1% of its original value. In math, this is the equation

$$\frac{Q}{Q(0)} = 0.01.$$

Plugging in the solution to the differential equation we have

$$\frac{Q(0)e^{-t/100}}{Q(0)} = 0.01.$$

The $Q(0)$ cancels, giving $e^{-t/100} = 0.01$. Solving for t gives about 460. What are the units??? Everything has been done in minutes, so this should be 460 minutes.

2. Page 59 #3

A tank originally contains 100 gal of fresh water. Then water contains 1/2 lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

2.1. **Hint.** This is precisely the same as the previous problem, except now we need 2 different differential equations. One governs the system ONLY until time t equals 10 minutes, and the other governs the system from then on. So just set up the first one, find the concentration after 10 minutes, and then use this concentration as the *initial* concentration for the next differential equation.

3. Page 60 #4

A tank with the capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

3.1. **Hint.** Oh no... here the volume of water is now changing, too, since we don't have a flow out that matches the flow in. Try to think about how to translate this information into mathematics! I think that the following equation should still hold:

$$\frac{dQ}{dt} = \text{rate}_{in} - \text{rate}_{out}.$$

The variable rate_{in} is still easy ((3 gal/min)(1 lb/gal)=3 lb/min). rate_{out} is a little harder. It should be $\text{rate}_{out} = \text{concentration}_{tank} \times \text{flowrate}_{out}$. But how do we find $\text{concentration}_{tank}$? This is just ($Q$ lb)/vol of water in gallons. Ah ha! The volume of water is changing as a function of time, however, since the flowrate in does not match the flowrate out. Well, this is a differential equation, too. It's just

$$\frac{d\text{vol}}{dt} = \text{flowrate}_{in} - \text{flowrate}_{out}.$$

The flowrates are constant, so just plugging in the numbers gives $\frac{d\text{vol}}{dt} = (3 - 2)$ gal/min. Solving this differential equation gives $\text{vol}(t) = t + C$, where C is some constant. What is C? Setting $t = 0$, we see that C is just the volume of the water at initial time, i.e. $C = \text{vol}(0) = 200\,\text{gal}$. Now plugging this into the stuff above gives

$$\text{concentration}_{tank} = Q/(t + 200).$$

So the final differential equation is just

$$\frac{dQ}{dt} = 3 - 2Q/(t + 200).$$

Now the question is how to solve this little guy.

Try using the method of integrating factors like in the last homework assignment (homework #6). This looks like a good candidate for that (and a good example of an application for all of that stuff that you stuck into your head).