HOMEWORK #6 PARTIAL SOLUTIONS (M427K FALL 2004)

1. Solve Differential Equation

$$xy' + (x^3 + x + 1)y = x^2$$

(hint: takes the form $y' + P(x)y = Q(x)$. Here $y' = \frac{dy}{dx}$)

2. Solve Differential Equation

$$y' + \sin(x)y = \sin(x)$$

(hint: same as previous exercise)

3. Solve Differential Equation

$$y' + e^x y = e^x + 3$$

(hint: same as previous exercise)

3.1. **Solution.** This equation takes the form $y' + P(x)y = Q(x)$ where $P(x) = e^x$ and $Q(x) = e^x + 3$, so let's figure out how to solve these types of equations in general (remember here $y'(x) \equiv \frac{dy}{dx}$).

This method will be called the "method of integrating factors". It will solve any equation of this form. Start by introducing our own function $\mu(x)$, and multiply both sides of the differential equation by it:

$$\mu y' + \mu P(x)y = \mu Q(x). \tag{3.1}$$

The reason for doing this is because we *hope* that equation 3.1 will look like

$$\frac{d(\mu y)}{dx} = \mu Q(x). \tag{3.2}$$

Well, how do we know that it *will* look like this? It will depend on how μ looks. But we invented μ, so we should be able to make it look right. Let's just expand out equation 3.2 using the product rule:

$$\mu y' + y\mu' = \mu Q(x).$$

Now compare this to equation 3.1. We want them to look the same. The RHS of both equations are certainly equal. The LHS has two terms. The $\mu y'$ term shows up in both equations as well (so they are equal). Hence, the second terms must be equal as well. In other words,

$$y\mu' = \mu P(x)y.$$

This is a differential equation (an easy one) that will give us the right formula for μ. Dividing both sides by y and solving for μ (this equation is separable... you could solve it the first day if you listened):

$$\mu(x) = e^{\int P(x)dx}. \tag{3.3}$$

So we have an expression for μ which makes equation 3.1 look like equation 3.2.

Hence, we only have to figure out how to solve equation 3.2. But that's easy... we just integrate:

$$\int d(\mu y) = \int \mu Q(x)dx + C.$$

(I put in the integration constant C so that you won't forget it into the final equations). Integrating the LHS gives

$$\mu y = C + \int \mu Q(x)dx.$$

Divide both sides by μ and plug in our equation that we found for μ (equation 3.3). This gives the final formula

$$y(x) = e^{-\int P(x)dx}\left\{C + \int e^{-\int P(x)dx}Q(x)dx\right\}.$$

This formula is general and is valid for all of these problems. The hard part now is just putting in the expressions for $P(x)$ and $Q(x)$ and actually INTEGRATING them.

I think that many of these cannot be integrated. Just do your best and write it in integral form if you can't figure out the integrals.

4. Solve Differential Equation

$$y' + \tan(x)y = \cos(x)$$

(hint: same as previous exercise)