HOMEWORK #4 (M427K FALL 2004)

1. Solve Differential Equation

$$(2x - 4y + 2)dx + (x - y)dy = 0$$

(hint: "kissing cousin" of the homogeneous equation. try substituting $x = \overline{x} + h$ and $y = \overline{y} + k$ to make it into a homogeneous equation. Then solve homogeneous equation as in Homework #2)

2. Solve Differential Equation

$$(2x - 4y + 2)dx - (6x - 8y + 1)dy = 0$$

(hint: same as previous exercise)

3. Solve Differential Equation

$$(x^2 + y)dx + (x + \sin y + e^y + 4^y)dy = 0$$

(hint: see if this takes the form $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$, i.e. check for exactness. Do you remember how to do this? Remember $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$)

3.1. **Solution.** This equation illustrates part of a larger idea that I don't expect you to fully understand right now.

Think for a minute about the differential equations that you have seen so far. They always come in the form $\text{blah}(x, y)dx + \text{blah}(x, y)dy = 0$, and you are basically trying to find y as a function of x such that if you plug in x and $y(x)$ then the differential equation is satisfied.

Usually you get some expression when you solve the differential equation that looks like $y^3 + xe^y + x^2 + \cos x = C$. The point is that it's a real mess, and in fact you cannot isolate x and y explicitly (you cannot write y as a function of x, right?). Nevertheless y and x determine each other in a complicated way. This is called an *implicit* equation (as opposed to an *explicit* equation where I can write down y as a function of x ... i.e. what you are used to). The surprise is that the graph of an implicit equation *still* traces out some curve in the xy-plane just like the graph of a normal (explicit) equation.

The point is that if this curve is "nice" enough, i.e. doesn't cross itself, doesn't kink, blah blah, then this curve looks *a lot* like a contour lines on a map (like a map of the mountains).

Have you ever looked at a contour map of the mountains before? The map itself is 2-dimensional (x and y coordinates) representing the surface of the earth, and the contours are just little circles and lines in the xy-plane that represent the lines of constant height (say 1000 feet, 1500 feet, 2000 feet, etc). So a contour is defined by some equation $F(x, y) = \text{constant} = C$ where $F(x, y)$ is the altitude at the point (x, y).

Now take the differential of both sides of the equation $F(x, y) = C$, which gives $dF(x, y) = 0$. This just says that the change of "height" $dF(x, y)$ is zero if I stay on the contour. That's obvious. I'm not walking up or down the mountain, but sideways across it.

On the other hand I can write down an expression for the change in height $dF(x, y)$ in terms of how quickly the height changes if I walk *only* in the x direction. This is written $\frac{\partial F}{\partial x}$. So if I walk only in the x direction then the change in height is given by how fast the height is changing multiplied by how far we go... i.e. $\frac{\partial F}{\partial x}dx$. I get a similar term if I walk in the y direction, so I have to add this on, giving the entire expression $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$.

So then the *differential equation* that traces out the contour is given by $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$. In the first term the y's are constant (we are moving ONLY in the x direction), and in the second term the x's are held constant (we are moving ONLY in the y direction). So we *could* just integrate the first term treating y as a constant, and we could integrate the second term treating x as a constant to recover the height $F(x, y)$.

Why in the hell would be want to do this? We already had an expression for $F(x, y)$. The point is that usually we don't. Usually we are just GIVEN the differential equation, and we want to know if the solution curves are going to look like the contour lines of SOME height function. If it does then our life is EASY, because we can just integrate as explained in the last paragraph.

But how do we check if the solution curves are going to look like some height contours unless we HAVE the solution already!!!??? The answer turns out to be surprisingly simple. I won't explain it to you, but essentially we only have to test something about the *second* derivatives. We need only to verify that $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$! It's true! So I just take the junk that's in front of the dx symbol (remember... I'm crossing my fingers and hoping that this looks like $\frac{\partial F}{\partial x}$), take the partial derivative $\frac{\partial}{\partial y}$ of it ("partial" means hold the x's constant), and compare it to the partial derivative $\frac{\partial}{\partial x}$ of the junk that is in front of the dy symbol. If they are the same, then BOOM!!! There IS some height function (you'll have to believe me), so now all I have to do is find it. This test is called testing for *exactness*.

So in this problem the differential equation I'm supposed to solve is

$$(x^2 + y)dx + (x + \sin y + e^y + 4^y)dy = 0.$$

Let's check if the solution is going to trace out the contours of a height function. So

$$\frac{\partial}{\partial y}(x^2 + y) = 1.$$

But

$$\frac{\partial}{\partial x}(x + \sin y + e^y + 4^y) = 1,$$

so they're both equal to 1 (and hence equal to each other). So this differential equation IS exact. So there IS a height function. Now let's just integrate the first term in the differential equation to find an expression for $F(x, y)$:

$$F(x, y) = \int_{y \text{ is constant}} \frac{\partial F}{\partial x} dx = \int_{y \text{ is constant}} (x^2 + y)dx.$$

This integral is just $\frac{x^3}{3} + yx + h$ where h is the constant of integration. But in this integral we specified that y is CONSTANT, so in fact h can be a function of y, i.e. $h(y)$.

On the other hand, we can integrate the second term in the differential equation to find a different expression for $F(x, y)$:

$$F(x,y) = \int_{y \text{ is constant}} \frac{\partial F}{\partial y} dy = \int_{y \text{ is constant}} (x + \sin y + e^y + 4^y) dy.$$

This integral is just $xy - \cos y + e^y + \frac{e^{y \ln 4}}{\ln 4} + g(x)$ where $g(x)$ is the constant of integration here.

Comparing these two expressions, we know that they have to be the same. So anything that is missing in one equation that is found in the other equation must show up in the unknown terms $h(y)$ and $g(x)$. So we obtain the solution $F(x, y) = C$ to our differential equation:

$$xy - \cos y + e^y + \frac{e^{y \ln 4}}{\ln 4} + \frac{x^3}{3} = C.$$