HOMEWORK #19 (M427K FALL 2004)

Introduction

This is to get you used to Taylor series. The point of Taylor series is to approximate complicated functions with things that we can understand (it's very important).

1. Expand $f(x)=\sin(x)$ to 1000 terms About 0

2. Expand $f(x) = \cos(x)$ to 1000 terms About 0

2.1. **Solution.** Just use the formula for the Taylor series:

$$f(x) = f(a) + \frac{f'(a)}{1!}x^1 + \frac{f''(a)}{2!}x^2 + \cdots + \frac{f^{(n)}(a)}{n!}x^n + \cdots.$$

In this case, we are expanding around $a = 0$, so we have

$$\cos(x) = \cos(0) + \frac{-\sin(0)}{1!}x^1 + \frac{-\cos(0)}{2!}x^2 + \frac{(-)(-)\sin(0)}{3!}x^3 + \cdots.$$

If we just clean this up (and use the fact that $\cos(0) = 1$ and $\sin(0) = 0$) then we see the pattern

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Now I'm supposed to expand the Taylor series to 1000 terms. Using the pattern above, I write it as a finite series plus a remainder:

$$\cos(x) = \sum_{n=0}^{500}(-1)^n \frac{x^{2n}}{(2n)!} + R_{1001}(x, \xi).$$

Here the remainder term has in it just the 1001st derivative evaluated at some point ξ somewhere between 0 and x, i.e.

$$R_{1001}(x, \xi) = \frac{f^{(1001)}(\xi)}{(n+1)!}x^{n+1}.$$

In this case, it is just the 1001st derivative of $\cos(x)$, so we have

$$R_{1001}(x, \xi) = \frac{-\sin(\xi)}{(n+1)!}x^{n+1}.$$

It is your job to find (for what values of x) where the remainder term goes to zero as $n \to \infty$. This expression was just done for $n = 1001$.

3. Expand $f(x) = e^x$ to 1000 terms About 0

4. Expand $f(x) = \frac{1}{1-x}$ to 1000 terms About 0