HOMEWORK #13 PARTIAL SOLUTIONS (M427K FALL 2004)

Introduction

This homework is meant for you to figure out what linear independence/dependence is.

1. Is This Linearly Independent or Dependent?

$$\{1, x, x^2, x^3, e^x\}$$

2. Is This Linearly Independent or Dependent?

$$\{\sin(x), \sin(2x)\}$$

2.1. **Solution.** Maybe the easiest way to attack this problem is to first take a guess at what YOU think, and then try to prove it. Note that, since I didn't write down a range for x, I am assuming x lies in *all* of the real numbers. I think that this is linearly INDEPENDENT, meaning that if I write down the equation

$$C_1 \sin(x) + C_2 \sin(2x) = 0,$$

then I think that the only solution is $C_1 = C_2 = 0$. In other words, I *don't* think that you can write one function in terms of the other(s). There are a couple of ways to prove it: I'll show you one (the other is the Wronskian method):

1) Trickery. I need to use some facts that I happen to know about sines. So let's use "proof by contradiction". In this method, I am assuming the *opposite* of what I want to show, then I show that this leads to ridiculous results. In other words, I am assuming that $C_1 \neq 0$ and/or $C_2 \neq 0$. So then this will lead to something bad, which will mean that my assumption was wrong (which will then mean that $C_1 = C_2 = 0$, which is what I *really* wanted to show).

Let's just use a double-angle identity on this bad boy:

$$C_1 \sin(x) + C_2 2 \sin(x) \cos(x) = 0$$

which implies

$$\sin(x) \{C_1 + 2C_2 \cos(x)\} = 0.$$

Hence either $\sin(x) = 0$ or $C_1 + 2C_2 \cos(x) = 0$. Since the first equation is NOT true (the sine does not vanish everywhere), we must assume the second equation. But we will show that this equation is also incorrect, meaning our initial assumption was incorrect.

So now assume that $C_1 + 2C_2 \cos(x) = 0$. Remember that I am assuming that NOT both C_1 and C_2 are zero. So then clearly $C_2 \neq 0$ because if it did then this equation would read $C_1 = 0$, which is exactly what I'm NOT assuming.

But then, since $C_2 \neq 0$ we can solve $\cos(x) = \frac{C_1}{2C_2}$. This means that the cosine is a constant for all x. But you KNOW that this isn't true either. It's all bogus.

So therefore our initial assumption was bogus (we assumed that NOT BOTH C_1 and C_2 vanish). Thus, in fact, $C_1 = C_2 = 0$, and these are linearly independent.

3. Is This Linearly Independent or Dependent?

$$\{\sin(x), \cos(x)\}$$

3.1. **Solution.** Again, I'll take a guess at it. I think that these are linearly INDEPENDENT. This time I'll prove it by the Wronskian method. Recall that if I have a set of functions $\{f_1, f_2, ..., f_n\}$ then the Wronskian is defined as

$$\det \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_2' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{\prime\cdots\prime} & f_2^{\prime\cdots\prime} & \cdots & f_n^{\prime\cdots\prime} \end{bmatrix}.$$

The test is that if this thing DOES NOT vanish EVERYWHERE, then this set is linearly independent. In our case, the Wronskian is just

$$\det \begin{bmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{bmatrix} = -\sin^2(x) - \cos^2(x) = -1 \neq 0,$$

so in our example the functions $\{\sin(x), \cos(x)\}$ are linearly independent. If you don't know how to take the determinant of a matrix, check on the internet or in a book. There are lots of ways to do it (all equivalent, of course). It's just a "nuts and bolts" type of calculation, and it's easy to do.

4. Is This Linearly Independent or Dependent?

$$\{e^{2x}, e^{3x}, e^{-x}\}$$

5. Is This Linearly Independent or Dependent?

$$\{e^x, xe^x\}$$