Solutions

M3150 PDEs midterm 2 (Spencer Stirling) - March 30, 2009

Directions: Use both the front and back of the paper for your solutions.

1) (2 points) Consider the 1-d heat equation on a rod of length L

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $u(x,t)$ is an unknown function. Determine (and SHOW YOUR WORK) if this equation is elliptic, hyperbolic, or parabolic.

General form $\quad Au_{xx} + 2Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = 0$

heat eqn $\quad c^2 u_{xx} + 0u_{xt} + 0u_{tt} + 0u_x + (-1)u_t + 0u = 0$

discriminant $\quad AC - B^2 > 0$ elliptic

$AC - B^2 < 0$ hyperbolic

$AC - B^2 = 0$ parabolic

for heat eqn $\quad AC - B^2 = 0 = \boxed{\text{parabolic}}$

2) (2 point) Consider the heat equation as above with the boundary conditions

$$u(0,t) \times u(L,t) = 1 \tag{2}$$

Is this PDE with these boundary conditions linear or nonlinear? If it is linear then determine if it is homogeneous or inhomogeneous (SHOW YOUR REASONING - guessing is worth nothing).

The PDE is linear. However the boundary conditions are of the form "$u^2 = 1$". This is nonlinear, hence the whole system is $\boxed{\text{nonlinear.}}$

3) Consider the problem of a vibrating string of length $L = 1$ meter. Assume that the string is made of a material such that the speed of wave propagation is $c = 1$ m/s. Furthermore assume that the ends of the string are tacked down and do not move. Suppose the string is initially deformed and can be described by the function

$$f(x) = \sin(\pi x) + \frac{1}{2}\sin(3\pi x) + 3\sin(7\pi x) \tag{3}$$

Furthermore suppose that the string has some initial velocity described by the function

$$g(x) = \sin(2\pi x) \tag{4}$$

a) (2 points) Write down the correct PDE with boundary/initial conditions that describes this problem

$u(x,t) \leftarrow$ unknown function

$$\frac{\partial^2 u}{\partial t^2} = \underset{\substack{\uparrow \\ 1}}{c^2} \frac{\partial^2 u}{\partial x^2}$$

b.c.

$u(0,t) = u(L,0) = 0$

initial conditions

$u(x,0) = f(x)$

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

b) (4 points) Solve the problem, i.e. find the solution $u(x,t)$ using the method of separation of variables.

$u(x,t) = X(x)\,T(t)$

plugging in $\quad X(x)\,T''(t) = c^2 X''(x)\,T(t) \Rightarrow \dfrac{T''(t)}{c^2 T(t)} = \dfrac{X''(x)}{X(x)} = k$

real constant

ODEs $\quad T''(t) - c^2 k\,T(t) = 0$

$X''(x) - k\,X(x) = 0$

case $k > 0 \quad (k = \mu^2)$

$X(x) = b_1 \sinh(\mu x) + b_2 \cosh(\mu x) \qquad$ b.c. $\Rightarrow b_1 = b_2 = 0$

case $k = 0$

$X''(x) = 0 \Rightarrow X(x) = b_1 x + b_2 \qquad$ b.c. $\Rightarrow b_1 = b_2 = 0$

place more work here

<u>case $k<0$</u> $\left(|k| = \mu^2\right)$

$$X''(x) + \mu^2 X(x) = 0$$

$$T''(t) + (\mu c)^2 T(t) = 0$$

so $X(x) = b_1 \cos(\mu x) + b_2 \sin(\mu x)$ b.c. $\Rightarrow b_1 = 0$

$\mu = \frac{n\pi}{L}$ $n = 1, 2, 3 \ldots$

<u>similarly</u> $T(t) = \tilde{b}_1 \cos(\mu c t) + \tilde{b}_2 \sin(\mu c t)$

$$u_n(x,t) = X(x)T(t) = b_1 \sin\left(\frac{n\pi}{L} x\right)\left\{\tilde{b}_1 \cos\left(\frac{n\pi}{L} c t\right) + \tilde{b}_2 \sin\left(\frac{n\pi}{L} c t\right)\right\}$$

absorb constants

$$\boxed{u_n(x,t) = \sin\left(\frac{n\pi}{L} x\right)\left\{\tilde{\tilde{b}}{}_1^n \cos\left(\frac{n\pi}{L} c t\right) + \tilde{\tilde{b}}{}_2^n \sin\left(\frac{n\pi}{L} c t\right)\right\}}$$

General solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

setting $L = 1$, $c = 1$ at $t = 0$ we have

$$u(x,0) = \sum_{n=1}^{\infty} \tilde{\tilde{b}}{}_1^n \sin(n\pi x) := f(x) = \sin(\pi x) + \frac{1}{2}\sin(3\pi x) + 3\sin(7\pi x)$$

so $\boxed{\tilde{\tilde{b}}{}_1^1 = 1, \quad \tilde{\tilde{b}}{}_1^3 = \frac{1}{2}, \quad \tilde{\tilde{b}}{}_1^7 = 3 \qquad \tilde{\tilde{b}}{}_1^n = 0 \text{ otherwise}}$

at $t = 0$ we take derivative $\frac{\partial}{\partial t}$:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} n\pi\, \tilde{\tilde{b}}{}_2^n \sin(n\pi x) := g(x) = \sin(2\pi x)$$

so $\boxed{\tilde{\tilde{b}}{}_2^2 = \frac{1}{2\pi} \qquad \tilde{\tilde{b}}{}_2^n = 0 \text{ otherwise}}$

4) Consider a 1-dimensional rod (a wire) of length $L = \pi$ meters that initially has a temperature distribution given by the function

$$f(x) = \begin{cases} 100 & 0 < x < \pi/2 \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

Assume that the ends of the wire are kept in a bath of antifreeze kept at 0° and that the thermal diffusivity is $c^2 = 1$.

a) (2 points) Write down the correct PDE with boundary/initial conditions that describes this problem

$u(x,t)$ = unknown fn

$$\frac{\partial u}{\partial t} = \underset{1}{c^2} \frac{\partial^2 u}{\partial x^2}$$

b.c. $u(0,t) = u(L,t) = 0$

initial cond $u(x,0) = f(x)$

b) (4 points) Solve the problem, i.e. find the solution $u(x,t)$ using the method of separation of variables.

$u(x,t) = X(x)T(t)$

plugging in $X(x)T'(t) = c^2 X''(x)T(t) \Rightarrow \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = k$

real const

case $k > 0$ ... trivial sol'n

case $k = 0$ trivial sol'n

case $k < 0$ $(|k| = \mu^2)$

$X''(x) + \mu^2 X(x) = 0 \Rightarrow X(x) = b_1 \cos(\mu x) + b_2 \sin(\mu x)$

apply b.c. $b_1 = 0$

$\mu = \frac{n\pi}{L}$

$T'(t) + (\mu c)^2 T(t) = 0 \Rightarrow T(t) = ce^{-(\mu c)^2 t}$

$u_n(x,t) = b_2 \sin\left(\frac{n\pi}{L} x\right) \cdot ce^{-\left(\frac{n\pi}{L} c\right)^2 t}$

place more work here

$$u_n(x,t) = b_n \sin\left(\frac{n\pi x}{\pi}\right) e^{-\left(\frac{n\pi}{\pi} c\right)^2 t} \quad c^2 = 1 \quad = b_n \sin(nx) e^{-(n^2)t}$$

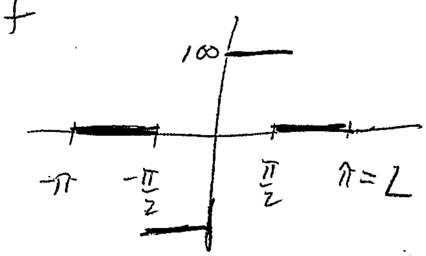
~~apply~~

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}}$$

apply initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) \equiv f(x) = \begin{cases} 100 & 0 < x < \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

half-range odd expansion for f



$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 100 \sin(nx)\,dx = -\frac{2\cdot 100}{\pi n}\cos(nx)\Big|_0^{\pi/2}$$

$$= \frac{200}{\pi n} \quad n \text{ odd}$$

$$\text{or } -\frac{200}{\pi n}(-1-1) \quad n = 2, 6, 10, 14, \ldots$$

$$\text{or } -\frac{200}{\pi n}(1-1) \quad n = 0, 4, 8, 12, \ldots$$

so $\boxed{b_n = \frac{200}{\pi n}}$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{\pi n} \sin(nx) e^{-n^2 t}$$

so $b_n = \dfrac{200}{\pi n}$ n odd

$b_n = \dfrac{400}{\pi n}$ $n = 2, 6, 10, 14, \ldots$

$b_n = 0$ $n = 0, 4, 8, 12, \ldots$

so $$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{-200}{n\pi}\left(\cos\left(\frac{\pi}{2}n\right) - 1\right)\sin(nx)\, e^{-n^2 t}}$$

5) (4 points) Consider a vibrating string of length $L = 10$ meters such that the end $x = 0$ is tacked down at a height 5 meters and the end $x = 10$ is tacked down at height 0 meters. Write down the appropriate PDE and boundary conditions (I did not specify an initial condition here). Find the *steady-state* solution $u(x,t)$ to this problem. (Remark: you have been finding steady-state solutions to the heat equation. This is NOT that situation. Note that a perfect (dissipationless) string will vibrate forever once set in motion - this differs from the heat equation where we know that the solution will approach a steady state as $t \to \infty$. In order to get to a steady state we would need to put an extra "friction term" into the PDE... or we can simply start in the steady state and remain there forever. Don't worry about the friction term for now).

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = 5 \qquad u(10,t) = 0}$$

steady state $\quad \frac{\partial u}{\partial t} = 0$

$$c^2 \frac{\partial^2 u}{\partial x^2} = 0 \implies u(x,t) = b_1 x + b_2$$

apply b.c. $\quad u(0,t) = 5 = b_2$

$$0 = u(10,t) = b_1 \cdot 10 + 5 \implies b_1 = -\frac{1}{2}$$

steady state $\quad \boxed{u(x,t) = -\frac{1}{2}x + 5}$