7.3 #12 Solution PDEs

solve $\dfrac{\partial u}{\partial t} + \sin t \dfrac{\partial u}{\partial x} = 0 \qquad u(x,0) = \sin x$

using Fourier transforms

Soln

$$\mathcal{F}_x\left(\frac{\partial u}{\partial t} + \sin t \frac{\partial u}{\partial x}\right) = \mathcal{F}(0) = 0$$

$$\|$$

$$\frac{\partial}{\partial t}\hat{u}(w,t) + \sin t\,(iw)\,\hat{u}(w,t) = 0$$

constant as far as x is concerned

$$\frac{\partial \hat{u}(w,t)}{\hat{u}(w,t)} = -iw \sin t \,\partial t$$

integrate both sides now

$$\ln\left(\hat{u}(w,t)\right) = +iw\cos t + \text{const}$$

so $\hat{u}(w,t) = C e^{iw\cos t}$

$u(x,0) = \sin x$ ⟵ this is not transformable since $\int_{-\infty}^{+\infty} |\sin x|\,dx = \infty$

but let's do it anyway (formally)

Breaking the LAW:

$$\hat{u}(\omega, 0) = \widehat{\sin}(\omega)$$

$\widehat{\sin}(\omega) = $ "$\mathcal{F}(\sin(x))(\omega)$

mysterious ill-defined beast

now $\widehat{\sin}(\omega) = \hat{u}(\omega, 0) = C e^{i\omega \overbrace{\cos 0}^{1}}$

$$\underline{\text{so}} \quad C = \frac{\widehat{\sin}(\omega)}{e^{i\omega}} = \widehat{\sin}(\omega) e^{-i\omega}$$

so $\hat{u}(\omega, t) = \widehat{\sin}(\omega) e^{i\omega(\cos t - 1)}$

so $u(x,t) = \mathcal{F}^{-1}\left(\widehat{\sin}(\omega) e^{i\omega(\cos t - 1)}\right)$

↓

7.3 # 12 pg 3

Now use shifting properties

$$\mathcal{F}^{-1}\left(e^{iaw}\hat{f}(w)\right) = f(x+a)$$

see 7.2 19, exercises 20

so $u(x,t) = \sin\left(x + (\cos t - 1)\right)$

(since for us here $a = \cos t - 1$ and $\hat{f} = \widehat{\sin}$

Now check the solution

plug into $\dfrac{\partial u}{\partial t} + \sin t \dfrac{\partial u}{\partial x} \overset{?}{=} 0$ ✓

also $u(x,0) = \sin x$ ✓

so the <u>method</u> was invalid, but the <u>result</u> is valid...

interesting...