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**Abelian Chern-Simons theory with toral
gauge group, modular tensor categories, and
group categories**

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group categories**

by

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Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

August 2008

To my family and friends

Acknowledgments

The author warmly thanks his advisor, Dan Freed, for suggesting this problem and for providing years of patient and persistent guidance during this work. The author also wishes to thank his wife, his family, and his friends for providing limitless moral support.

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August 2008

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Publication No. _____

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The University of Texas at Austin, 2008

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Classical and quantum Chern-Simons with gauge group $U(1)^N$ were classified by Belov and Moore in [BM05]. They studied both ordinary topological quantum field theories as well as spin theories. On the other hand a correspondence is well known between *non-spin* $(2 + 1)$ -dimensional TQFTs and modular tensor categories. We study group categories and extend them slightly to produce modular tensor categories that correspond to toral Chern-Simons. Group categories have been widely studied in other contexts in the literature

[FK93],[Qui99],[JS93],[ENO05],[DGNO07]. The main result is a proof that the associated projective representation of the mapping class group is isomorphic to the one from toral Chern-Simons. We also remark on an algebraic theorem of Nikulin that is used in this paper.

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Chapter 1

Introduction

The study of topological quantum field theories emerged in the 1980's in [Wit88] where a supersymmetric quantum theory was introduced that is linked to Floer homology and the Donaldson invariants. It was shown that this quantum field theory is metric independent. A short time later groundbreaking connections were made in [Wit89] between Chern-Simons field theory and low-dimensional topology (knot theory and 3-manifold invariants).

Contemporarily, algebraists and representation theorists were constructing *quantum groups*, and equally powerful connections were made between quantum groups, knot theory, and 3-manifold invariants ([RT90],[RT91], [KM91]).

In Chern-Simons theory the basic data that characterizes a theory is a compact Lie group G along with an element $k \in H^4(BG, \mathbb{Z})$ called the *level*. Witten considered compact semisimple Lie groups (e.g. $SU(2)$) where k is an integer. On the other hand the basic data that characterizes a quantum group is a compact semisimple Lie group G along with a deformation parameter t .

It was noticed immediately that there is an agreement between Chern-Simons theory and quantum groups when comparing the induced link invari-

ants and 3-manifold invariants. For example, for $G = \text{SU}(2)$ they agree if the level and the deformation parameter are related by

$$t = \exp\left(\frac{\pi i}{2(k+2)}\right) \tag{1.1}$$

In light of this (actually somewhat before) Atiyah proposed an axiomatic umbrella formulation of TQFTs [Ati90a] that unifies both approaches into a common language.

Simultaneously a third line of development based on category theory was emerging. The braided and ribbon categories described in [JS93],[Shu94], combined with aspects formulated in [MS89],[RT90],[RT91], resulted in *modular tensor categories* (c.f. [Tur94]). In particular quantum groups are examples of modular tensor categories, and many crucial aspects of conformal field theory are also encoded in modular tensor categories. By the early 1990's a clearer picture had emerged:

$$\text{Quantum Groups} \subset \text{MTCs} \iff (2+1)\text{-dim TQFTs} \supset \text{Chern-Simons} \tag{1.2}$$

The relationship between MTCs and TQFTs is discussed further below (in particular - to the author's knowledge - the direction $\text{MTC} \leftarrow \text{TQFT}$ is not yet constructed for *all* cases).

Several examples of Chern-Simons theories that have been quantized are listed in the left column of table (1.1) (more cases that have been quantized include most simple groups G and direct products). In particular Chern-Simons theories with gauge group $\text{U}(1)$ were studied by Manoliu [Man98] using a real polarization technique, and more recently Chern-Simons theories with

gauge group $U(1)^N$ were classified by Belov and Moore [BM05] using Kähler quantization. It was shown that the data that determines quantum toral Chern-Simons is a trio (\mathcal{D}, q, c) where \mathcal{D} is a finite abelian group, $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ is a quadratic form, and c is an integer mod 24 (subject to a constraint). It is natural to ask what the corresponding modular tensor categories are. This paper answers that question.

Table 1.1: Zoology

Classical Chern-Simons G compact Lie group $k \in H^4(BG, \mathbb{Z})$	Modular Tensor Category	Link Invariants in S^3 (link has canonical framing)
$G = \mathrm{SU}(2)$ $k \in \mathbb{Z}$	Quantum group $U_t(\mathfrak{sl}_2(\mathbb{C}))$ $t = \exp(\frac{\pi i}{2(k+2)})$	Jones polynomial
$G = \mathrm{SU}(N)$ $k \in \mathbb{Z}$	Quantum group $U_t(\mathfrak{sl}_N(\mathbb{C}))$ $t = \exp(\frac{\pi i}{2(k+2)})$	HOMFLY polynomial
$G = \text{finite group}$ $k = \text{vacuous}$ [FQ93] [DW90]	Quantum double $\mathcal{D}(G)$	No uniform description
$G = U(1)$ [Man98] $G = U(1)^N$ [BM05] $k = \text{even lattice}$	this paper	Deloup invariants [Del99]

We note that Belov and Moore [BM05] classified more general *spin*¹ toral Chern-Simons theories as well. Unfortunately there is no well-developed notion of *spin modular tensor category*, however the work done here makes an excellent toy model that we can use to decide what the “right” definition for spin MTC should be. We plan to expand these ideas in a forthcoming paper.

¹The ordinary theories below are constructed on manifolds with extra structure: 2-framings [Ati90b]. *Spin* theories are really theories of *framed* manifolds. See the recent work by Hopkins-Lurie on the Baez-Dolan hypothesis.

Physicists will be mainly interested in the applications to the fractional quantum Hall effect (FQHE). The *abelian* states at filling fraction $\nu = \frac{1}{3}$ remain the only rigorously-established experimental states to coincide with Chern-Simons, hence the abelian case remains relevant despite being useless for topological quantum computation.

Before we proceed let us mention the very closely related work of Deloup [Del99, Del01, Del03]. Deloup begins with the data of a finite abelian group \mathcal{D} and a quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$. Because of the abelian nature of the data it is possible to construct invariants of links and (eventually) a $(2 + 1)$ -dim TQFT “by hand” appealing to reciprocity alone.

This bypasses modular tensor categories entirely. However, the price is that no braiding is described (the braiding is rather more subtle than what one might first expect). We emphasize this difference since the modular tensor categories described here allow us to construct an *extended* $(2 + 1)$ -dim TQFT (see chapter (2)). In particular we can describe quasiparticles completely, whereas Deloup’s theories cannot. We also emphasize that Deloup did not connect his work to Chern-Simons. It is the main result in this paper that the TQFTs constructed here are the same as those from toral Chern-Simons.

Finally, some simple examples of ribbon categories are considered in the appendix in [Del99]. These examples are briefly considered here in chapter 4, and we argue that these do *not* correspond to toral Chern-Simons since many of them are not modular tensor categories.

The organization of this paper is as follows: in chapter (2) we give a brief overview of TQFTs starting with the motivating example of Chern-Simons. In chapter (3) we review toral Chern-Simons as was classified by Belov-Moore. In chapter (4) we provide the relevant definitions for ribbon

categories and modular tensor categories, and we construct $(2+1)$ -dim TQFTs from them. This chapter differs from [Tur94] and [BK00] in that we emphasize *non-strict* categories. In chapter (5) we study group categories and build modular tensor categories out of them. The main result is proven in chapter (6) - the projective actions of the mapping class group induced from toral Chern-Simons and separately from group categories are isomorphic.

Chapter 2

$(2 + 1)$ -dim Topological Quantum Field Theories

2.1 Introduction

In this chapter we give a quick account of $(2 + 1)$ -dim topological quantum field theories (TQFTs). A $(2 + 1)$ -dim Chern-Simons TQFT is essentially determined by the $(1 + 1)$ -dim conformal field theory (CFT) on the boundary (the Knizhnik-Zamolodchikov equations determine the braidings and the twists that appear in the theory). The language of modular tensor categories (MTCs) is rather different, but underneath the details MTCs axiomatically encode the relevant structures that appear in CFTs. Hence it is no surprise that the Chern-Simons TQFTs form a subset of the TQFTs constructed from modular tensor categories (it is in debate whether the opposite inclusion is true [HRW07]).

The axiomatic approach toward the end of the chapter is taken from

chapter 3 in Turaev’s book [Tur94] as well as the book of Bakalov and Kirillov [BK00]. The original axioms were formulated by Atiyah [Ati90a] long ago.

Witten’s work relies on the earlier work of Segal [Seg04] and Moore and Seiberg [MS89] in conformal field theory. Briefly, the conformal field theory that appears on the boundary is the Wess-Zumino-Witten (WZW) model (actually the chiral/holomorphic part). For a geometric perspective on CFTs and Chern-Simons we recommend [Koh02].

Although Atiyah’s axioms apply in any dimension, we wish to restrict ourselves to $(2+1)$ -dimensions. In this case all of the known examples are considerably richer than Atiyah’s axioms might suggest. Framed links (ribbons) appear that physically are meant to encode the worldlines of exotic *anyonic quasiparticles* [Wil90] undergoing creation, annihilation, twisting, and braiding.¹ Mathematically more general colored ribbon graphs are studied, and *surgery* provides a route from the ribbon graph construction to Atiyah’s axioms.²

2.2 Chern-Simons

In [Wit89] Witten studied the Chern-Simons quantum field theory defined by the action

$$\exp(2\pi i k S_{CS}) = \exp\left(2\pi i k \int_{X_3} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)\right) \quad (2.1)$$

We will discuss such actions more coherently in chapter (3), but for now X_3 is a compact oriented 3-manifold equipped with a vector bundle $E \rightarrow X$ with

¹the ribbons must be “colored” with the particle species.

²we note that all manifolds must be *oriented* throughout this paper.

structure group G . For now G is a connected and simply-connected compact Lie group (e.g. $SU(2)$). Chern-Simons is defined to be a gauge theory where the gauge group is $\mathcal{G} = \text{Map}(X_3, G)$, i.e. configurations of A that are related by gauge transformations are physically indistinguishable and must be identified.

The action as written is not always well-defined since A is not always well-defined. However, obstruction theory tells us that E is trivializable for a connected simply-connected compact Lie group G (this is *not* true in general, nor even in the remainder of this paper). Once we choose a trivialization³ then this determines a standard flat covariant derivative D^0 (the trivialization determines parallel transport). Then given any other covariant derivative D we can define the vector potential A via $D = D^0 + A$.⁴

It is well known (see e.g. [Fre95]) that the integral in equation (2.1) is *not* gauge invariant. However under gauge transformations (if X_3 is closed) the integral changes by integer values M only:

$$\int_{X_3} \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \rightarrow \int_{X_3} \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) + M \quad (2.2)$$

Hence $\exp(2\pi i k S_{CS})$ is *invariant* as long as the *level* k is any arbitrary integer.

More generally if X_3 has boundary Σ_2 then under gauge transformations the integral instead picks up a chiral Wess-Zumino-Witten term:

$$\int_{X_3} \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \rightarrow \int_{X_3} \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) + S_{cWZW} \quad (2.3)$$

³This choice of trivialization is unimportant. Any two trivializations are related by a gauge transformation, but we shall physically identify any two connections that are gauge equivalent. Hence we only require that the bundle be *trivializable*.

⁴ E has structure group G , and D must respect this (e.g. parallel transport takes orthonormal frames to orthonormal frames for $G = \text{SO}(3)$). Thus, A is valued in the Lie algebra \mathfrak{g} .

It is a fact that S_{cWZW} depends only on the configuration on the boundary Σ_2 . Clearly the action is *not* gauge invariant if interpreted in the usual sense. However the WZW term satisfies crucial *cocycle conditions*, and a more formal construction yields a gauge invariant theory (see pgs. 16-21 in [Fre95]).

It is instructive to consider briefly a physical system that Chern-Simons is thought to describe. In the fractional quantum Hall effect (FQHE) a 2-dimensional gas of electrons (trapped between semiconductor layers) is cooled to a few milliKelvin and placed under a magnetic field pointing in the z direction (if the 2-d gas lies in the xy -plane). Schematically the action is

$$S := S_{\text{cyclotron}} + S_{\text{e-e interaction}} \quad (2.4)$$

where the cyclotron term describes the electrons orbiting in circular paths due to the magnetic field, and the interaction term describes Coulomb repulsion between electrons. The magnetic field breaks the parity reversal symmetry of the system - hence the system is *chiral*. Consider the 2-d electron gas propagating in time. Then this is a $(2 + 1)$ -dimensional classical field theory.

Ignoring the e-e interaction term momentarily the quantum description is given in terms of (degenerate) Landau levels where the energy of the N th level goes as $E_N \sim \sqrt{N}$. Hence the system is gapped, and sufficiently lowering the temperature restricts the system to the degenerate ground state $N = 0$.

The ground Landau level obtains interesting structure when e-e interactions are again considered. It is shown in [HLR93] that the action (through a change of variables) can be written as the effective action

$$S := S_{\text{cyclotron}} + S_{\text{e-e interaction}} \xrightarrow{N=0} S_{CS} + S_{\text{quasiparticles}} \quad (2.5)$$

where S_{CS} is the Chern-Simons action introduced in equation (2.1), and $S_{\text{quasiparticles}}$ is a term encoding the dynamics of exotic *anyonic quasiparticles* [Wil90]. The quasiparticles can be viewed as quantum excitations of cooperating electrons and magnetic flux quanta [Kha05]. However, we always treat them semiclassically in the sense that their *trajectories* are treated as *classical* paths. ⁵

The quasiparticles are coupled to A , hence they can be viewed as *detectors* that measure the properties of A . We can imagine quasiparticle/antiquasiparticle pairs being created, possibly braiding around each other, and annihilating. Then their worldlines form links in $(2+1)$ -dimensions. Furthermore each quasiparticle species has a $(2+1)$ -dimensional analogue of spin - the *twist* - which is a phase factor that a quasiparticle picks up when it is spun one full counterclockwise turn (viewing the xy -plane from above). Hence the worldlines should be thought of as *framed* links, or ribbons, to encode the twists.

As a first attempt to understand the role of the quasiparticles let us alter the classical setup slightly. Instead of a Hamiltonian scenario where the 2-d electron gas propagates forward in time (i.e. a 3-manifold of the form $\Sigma \times I$), suppose we have a *closed* compact oriented 3-manifold X_3 with a fixed vector potential A . Although we are in a classical setting we put in *by hand* quasiparticles (which are quantum-mechanical). However, as already mentioned we only allow classical trajectories, and we treat them only as detectors to measure aspects of A . We also ignore the possible twisting of the quasiparticles (this will be remedied later).

Then the creation and annihilation of a quasiparticle/antiquasiparticle

⁵Note that the Chern-Simons vector potential A is usually *not* the vector potential associated to the magnetic field.

pair forms a simple closed curve γ in X_3 . The quasiparticle is labeled by a representation R of G , and the antiquasiparticle is labeled by the dual representation R^* (the appearance of representations is consistent since the quasiparticles are quantum mechanical objects put in by hand). The measured observable is defined to be the *Wilson loop*

$$W_R(\gamma, A) := \text{Tr}_R \text{Hol}_\gamma(A) \quad (2.6)$$

the trace of the holonomy of A around γ in the representation R .

Now we wish to treat A quantum mechanically (leaving the quasiparticles in their semiclassical *detector* roles). It is clearer if we use the path integral perspective. Then the quantum observable associated to a simple closed curve γ colored with the species R is a *weighted average* of $W_R(\gamma, A)$ over all configurations of A :

$$\langle W_R(\gamma, A) \rangle := \int_{\mathcal{A}} \mathcal{D}\mathcal{A} \exp(2\pi i k S_{CS}) W_R(\gamma, A) \quad (2.7)$$

It is easy to generalize this to multiple link components with different colorings R_i . In the absence of link components we obtain a 3-manifold invariant of X_3 :

$$Z(X_3) := \int_{\mathcal{A}} \mathcal{D}\mathcal{A} \exp(2\pi i k S_{CS}) \quad (2.8)$$

Unfortunately the path integral quantization procedure is not typically calculable, is not rigorously defined, and the quasiparticles have rather limited properties in this formulation (i.e. no twisting and no fusing into composite quasiparticles). Furthermore we have restricted ourselves to closed 3-manifolds. The ultimate remedy is a Hamiltonian quantization procedure

involving Kähler quantization (no quasiparticles) and conformal field theory (includes quasiparticles) which we briefly discuss now.

Phase space

Now let us briefly recall some aspects of the phase space described in [Wit89]. It is simplest to first consider a theory on $\Sigma \times I$ where Σ is a closed oriented 2-surface *without marked arcs*.⁶ As usual the canonical formalism begins by describing the space of configurations on the initial time surface $\Sigma \times \{0\}$. For Chern-Simons the initial configuration is a smooth Lie algebra-valued vector potential A_2 (a field configuration) on $\Sigma \times \{0\}$.⁷

Given an arbitrary field configuration A_2 on the initial time slice $\Sigma \times \{0\}$ (subject to the appropriate constraints) we can use the equations of motion to propagate it forward in time (producing a vector potential configuration A on the whole 3-manifold).⁸ In this way the different “configuration spaces” at different time slices $\Sigma \times \{t_1\}$ and $\Sigma \times \{t_2\}$ can be identified and we need only think of *the* configuration space. On the other hand the resulting connection A on the entire 3-manifold $\Sigma \times I$ (by construction) is a solution to the Euler-Lagrange equations, hence alternatively we can view the “configuration space” as the space of solutions to the Euler-Lagrange equations on the 3-manifold.

⁶Arcs become ribbons when propagated in time - these are the worldlines of quasiparticles. The marking (coloring) is the particle species.

⁷We note that a given A_2 configuration on the initial time surface cannot be completely arbitrary because for some vector potentials we would have no hope of solving forward to produce a solution of the Euler-Lagrange equations on the whole 3-manifold. Hence we can only consider vector potentials on the 2-surface that are subject to the *Gauss law constraint*.

⁸Usually it is necessary to specify the initial field configuration *and* time derivative(s) in order to solve forward using the equations of motion (since typically Euler-Lagrange equations are second-order differential equations). However (as we shall see) the Euler-Lagrange equations are first-order for Chern-Simons, hence the time derivatives are not necessary.

For Chern-Simons the Euler-Lagrange equation

$$F = 0 \tag{2.9}$$

says that classically the allowed connections on the 3-manifold $\Sigma \times I$ must be flat. Hence naively the configuration space should be the space $\mathcal{A}^{F=0}$ of flat vector potentials A on $\Sigma \times I$.

However Chern-Simons has in addition the assumed mathematical redundancy that defines it as a gauge theory, so instead the configuration space is the space of flat vector potentials on $\Sigma \times I$ modulo gauge transformations, the *moduli space of flat connections*

$$\mathcal{M} := \mathcal{A}^{F=0} / \sim \tag{2.10}$$

Alternatively, we can work over the initial time slice $\Sigma \times \{0\}$ and consider the space \mathcal{A}_2 of vector potentials over the 2-manifold that satisfy the Gauss law constraint. For Chern-Simons the Gauss constraint is easy - the curvature of an allowed configuration A_2 over $\Sigma \times \{0\}$ must vanish, i.e. $F_2 = 0$. When restricting a vector potential A on the 3-manifold to the initial time slice $\Sigma \times \{0\}$ we must use up part of the gauge freedom in order to kill the time component of the 1-form A . This is *temporal gauge*.

Even in temporal gauge there is still gauge freedom left. Modding out by this residual gauge freedom we obtain the configuration space, again called the *moduli space of flat connections*

$$\mathcal{M} := \mathcal{A}_2^{F_2=0} / \sim_2 \tag{2.11}$$

We will freely switch back and forth between the two definitions of configuration space.

Example 2.12. \mathcal{M} was studied in [AB83] and [Jef05], however there it arises from Yang-Mills theory on a 2-dimensional oriented surface Σ *with Riemannian metric*.⁹

Since it will be useful later let us remind ourselves of some elementary facts about Riemann surfaces (see e.g. [Sch89]). For an oriented 2-surface Σ the metric induces a unique complex structure.¹⁰ Conversely, the uniformization theorem says that a complex structure on a 2-surface Σ induces an orientation and a *class* of metrics that are all equivalent up to local conformal transformations (angles are preserved, but not necessarily lengths). One of those has normalized constant scalar curvature.¹¹

Hence for an orientable 2-surface we have a one-to-one correspondence

$$\text{complex structures} \leftrightarrow \text{conformal classes of metrics and orientations} \quad (2.13)$$

Since in Yang-Mills Σ is endowed with a Riemannian metric we might as well give Σ the induced complex structure.

Let $E_2 \rightarrow \Sigma$ be a vector bundle with structure group G on which Yang-

⁹ Σ must have a metric because the Hodge star ($*$) operator is used in the Yang-Mills action.

¹⁰One can define an *almost* complex structure J via the following map: for a tangent vector ξ , $J(\xi)$ is the unique vector that is

1. the same length as ξ ,
2. orthogonal to ξ ,
3. the pair $(\xi, J(\xi))$ has positive orientation.

Any “almost” complex structure on a surface is integrable, so this is actually a complex structure.

¹¹normalized to -1 , 0 , or 1

Mills lives. It is straightforward to show that *if the vector bundle E_2 is trivial* then the 2-dimensional Yang-Mills equations of motion are

$$F_2 = 0 \tag{2.14}$$

Modding out by gauge transformations we recover the moduli space of flat connections \mathcal{M} .

However a *flat* G -connection corresponds to a homomorphism

$$\pi_1(\Sigma) \rightarrow G \tag{2.15}$$

since a connection can be encoded as monodromies along paths (E_2 is a *trivial bundle* so that the monodromy along a non-closed path makes sense). Two paths that start at a point x_1 and end at a point x_2 form a loop, and the difference in monodromies is just the holonomy around the loop. However, the holonomy of a *flat* connection around a contractible loop is always the identity. Using this it is easy to show that a homotopy of a non-closed path (leaving the endpoints fixed) leaves the monodromy invariant. Hence the space of flat G -connections (even before modding out by gauge transformations) is determined by the holonomies around generators of $\pi_1(\Sigma)$.

As a very easy example consider $\Sigma = S^2$. Then $\pi_1(S^2) = 0$ hence there is only the trivial homomorphism $\pi_1(\Sigma) \rightarrow G$. Thus there is only a single flat connection, so \mathcal{M} is just a point. In particular we see that \mathcal{M} is *compact* and even-dimensional; these are features that persist for general Σ .¹²

¹²Here we can see that the assumed triviality of the vector bundle E_2 is essential. If not then we could consider the example $\Sigma = S^2$ and take as the vector bundle the tangent bundle TS^2 . Give S^2 a metric (say a metric of constant curvature 1 by thinking of S^2 as standardly embedded in \mathbb{R}^3). Then the tangent bundle is an $SO(2)$ -bundle. Since $\pi_1(S^2) = 0$ we might

In other theories the Euler-Lagrange equations are typically second order differential equations. In the canonical formalism it is customary to formally pass to a first-order theory at the cost of adding extra *momentum* variables. At the initial time slice $\Sigma \times \{0\}$ the *phase space* is the space of allowed positions *and* momenta, and we propagate this phase space forward to any other time slice using Hamilton's equations.

In Chern-Simons, however, the Euler-Lagrange equations are already first-order differential equations. Thus it is inappropriate to introduce auxiliary canonical momenta (any attempt to do so will yield a constrained mechanical system where the momenta Π can be written in terms of the configuration variables A). Hence the moduli space of flat connections \mathcal{M} (in addition to being the configuration space) also plays the role of phase space equipped with a symplectic structure and a Hamiltonian.

Let us remark briefly about the origin of the symplectic structure on \mathcal{M} . We refer the reader to [Jef05] for more details. First, in order to be a symplectic manifold we need that \mathcal{M} is even dimensional. Given the identification above of a flat connection with a homomorphism

$$\pi_1(\Sigma) \rightarrow G \tag{2.16}$$

the dimension of \mathcal{M} is $2g \cdot \dim G$ where g is the genus of Σ , hence manifestly the dimension is even.

Second, consider the space of all G -connections $\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g})$ over the

conclude by the argument above that the tangent bundle admits a unique flat connection. However, the Gauss-Bonnet theorem implies that no flat connection exists on TS^2 since the Euler characteristic is $\chi(S^2) = 2$ whereas the integral of a flat connection is just 0. The resolution is that TS^2 is not trivial.

2-manifold Σ . Since \mathcal{A} is an affine space (actually here it is a vector space because there is a distinguished $A = 0$ corresponding to the chosen standard flat connection D^0), each tangent space $T\mathcal{A}_A$ can be identified with \mathcal{A} itself. Hence a symplectic form on the *manifold* \mathcal{A} is determined by a symplectic form on the *vector space* \mathcal{A} . A natural symplectic form is given by (up to normalization)

$$\omega(A_1, A_2) = \int_{\Sigma} \text{Tr } A_1 \wedge A_2 \quad (2.17)$$

We leave it to the references for proof that these statements descend to \mathcal{M} .

Prequantization

We turn our attention towards quantization of the *compact* symplectic phase space \mathcal{M} . However, we should expect difficulties since in other theories typically phase space is non-compact.

Since \mathcal{M} also plays the role of configuration space we might try to make sense of $L^2(\mathcal{M}, \mathbb{C})$.¹³ Indeed if $\dim(\mathcal{M}) = 2n$ then we have the usual Liouville volume form

$$\text{vol} = (-1)^{n(n-1)/2} \frac{1}{n!} \omega \wedge \omega \wedge \dots \wedge \omega \quad (2.18)$$

where the wedge product is over n copies of the symplectic form ω .

Hence we know how to integrate functions on \mathcal{M} , so $L^2(\mathcal{M}, \mathbb{C})$ is well-defined. Intuitively the number of quantum basis wavefunctions should be proportional to the volume (a quantum basis state corresponds to a box of side \hbar in phase space). Since \mathcal{M} is compact the total volume of \mathcal{M} is finite, hence we expect finitely-many quantum basis wavefunctions. Unfortunately,

¹³We feel that this would be an interesting problem to compare in this context using *spin networks*. See for example [Bae96] and [Bae99].

even though \mathcal{M} is compact, $L^2(\mathcal{M}, \mathbb{C})$ is infinite dimensional.¹⁴ Therefore we assert that $L^2(\mathcal{M}, \mathbb{C})$ is too large to describe the quantum states.

The technique of geometric quantization [Woo80] provides a more rigorous quantization that agrees with our intuition. We briefly describe the main ideas.

Instead of $L^2(\mathcal{M}, \mathbb{C})$ we can consider L^2 sections of a hermitian line bundle \mathcal{L} (a $U(1)$ bundle equipped with a $U(1)$ covariant derivative ∇) over \mathcal{M} . Denote the space of these L^2 sections $L^2(\mathcal{L})$. We refer the reader to pgs 16-18 of [Fre95] for the construction of \mathcal{L} from the Wess-Zumino-Witten model.

$L^2(\mathcal{L})$ is the *prequantum Hilbert space*. Unfortunately (exactly as is the case for $L^2(\mathcal{M}, \mathbb{C})$) $L^2(\mathcal{L})$ is infinite dimensional. In order to shrink to a finite-dimensional physical Hilbert space it is instructive to recall that \mathcal{M} *also* plays the role of phase space. In this light $L^2(\mathcal{L})$ is too large since it is analogous to “ $L^2(p, q)$ ”, i.e. L^2 functions on both the position and momentum variables.

Kähler quantization

Choosing a polarization is the process of picking a foliation of \mathcal{M} by leaves that are precisely half the dimension of \mathcal{M} . At a point $x \in \mathcal{M}$ the leaf P that passes through x determines locally a “momentum” submanifold of \mathcal{M} . The physical Hilbert space is defined to be the subspace of $L^2(\mathcal{L})$ of sections that are *constant* in the momentum direction.

More precisely at $x \in P \subset \mathcal{M}$ the tangent space $TP_x \subset T\mathcal{M}_x$ must be a

¹⁴For example Fourier series provides a countably-infinite basis for functions on the compact manifold S^1 .

Lagrangian subspace (maximal isotropic) with respect to the symplectic form ω , i.e. TP_x is an n -dimensional subspace (\mathcal{M} is $2n$ dimensional) such that if $A_1, A_2 \in TP_x$ then $\omega(A_1, A_2) = 0$. The physical Hilbert space is comprised of sections s such that $\nabla_A s = 0$ for every $A \in \Gamma(TP)$. There are several methods for choosing a polarization, however each requires that we impose *extra structure* on \mathcal{M} .

We now describe a similar method for reducing the phase space. The idea is to equip \mathcal{M} with a complex structure J and restrict to *holomorphic sections*. For technical reasons it is useful if \mathcal{M} can be made Kähler. We already have a symplectic form (possibly not normalized properly)

$$\omega(A_1, A_2) = \int_{\Sigma} \text{Tr } A_1 \wedge A_2 \quad (2.19)$$

and a choice of complex structure J . Then \mathcal{M} is Kähler if we define the Riemannian metric $g(A_1, A_2)$ to be

$$g(A_1, A_2) = \omega(A_1, J \cdot A_2) \quad (2.20)$$

Now shrink the prequantum Hilbert space using standard complex analysis: an almost complex structure is a (fiberwise) linear map $J : T\mathcal{M} \rightarrow T\mathcal{M}$ that satisfies $J^2 = -1$. $T\mathcal{M}$ is a real vector bundle, but over the reals J has no eigenvalues. However, if we complexify $T\mathcal{M}$ (which doubles the real dimension) then $T\mathcal{M}_{\mathbb{C}}$ splits into $\pm i$ eigenspaces of J , i.e.

$$T\mathcal{M}_{\mathbb{C}} = T\mathcal{M}^{(1,0)} \oplus T\mathcal{M}^{(0,1)} \quad (2.21)$$

(we should also complexify the symplectic form $\omega_{\mathbb{C}}$ and the covariant derivative

$\nabla^{\mathbb{C}}$ in the line bundle \mathcal{L}). Then the *holomorphic sections* of \mathcal{L} are sections s such that $\nabla_A^{\mathbb{C}} s = 0$ for every $A \in \Gamma(T\mathcal{M}^{(0,1)})$. Define the physical Hilbert space \mathcal{H} to be the space of holomorphic sections of \mathcal{L} .

Extra assumption: complex structure on Σ

The only issue left to resolve is the *choice* of complex structure J on \mathcal{M} . However, recall that \mathcal{M} is the moduli space of flat connections on Σ .

Let us equip Σ with a Riemannian metric. Then there is an induced natural complex structure J on the manifold $\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g})$ that can be seen as follows. Since \mathcal{A} is an affine space (actually a vector space because of the distinguished $A = 0$ due to a choice of standard flat connection D^0) the tangent space $T_A \mathcal{A}$ at a point $A \in \mathcal{A}$ can be identified with the vector space \mathcal{A} itself. Hence a complex structure on the manifold \mathcal{A} is determined by a linear operator J acting on the vector space \mathcal{A} such that $J^2 = -1$. Such a map is given by

$$J(A) = *A \tag{2.22}$$

where $*$ is the Hodge dual. Because Σ is 2-dimensional it is trivial to verify that $J^2 = (*)^2 = -1$ on 1-forms - so this defines a complex structure on \mathcal{A} (which descends to a complex structure on the moduli space \mathcal{M}).

The symplectic form is

$$\omega(A_1, A_2) = \int_{\Sigma} \text{Tr } A_1 \wedge A_2 \tag{2.23}$$

and the complex structure ¹⁵ is defined by

$$J(A) := *A \tag{2.24}$$

Hence a Kähler structure on \mathcal{A} is achieved by using the Riemannian metric

$$g(A_1, A_2) = \omega(A_1, J \cdot A_2) = \int_{\Sigma} \text{Tr } A_1 \wedge *A_2 \tag{2.25}$$

Passing to moduli space we obtain a Kähler structure on \mathcal{M} .

Example 2.26. We note that the full strength of a Riemannian metric on Σ is not required to produce the complex structure J on \mathcal{M} .

Recall from example (2.12) that a given orientation and Riemannian metric on a 2-surface Σ induces a complex structure j on Σ (see below in local coordinates). However, let us forget the Riemannian metric on Σ and *start* with a complex structure j on Σ . Then j induces a complex structure J' on the affine manifold $\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g})$ (since each tangent space $T_A\mathcal{A}$ is identified with the vector space \mathcal{A} itself). Passing to the moduli space we obtain a complex structure J' on \mathcal{M} .

In local coordinates it is straightforward to see that J' is actually the *opposite* complex structure to the J defined using a Riemannian metric on Σ and the Hodge star operator (see [GH78] for the relevant complex geometry).

For example consider the 2-dimensional plane \mathbb{R}^2 equipped with the standard inner product and standard orientation. Let us ignore the fact that

¹⁵Again we ignore integrability of this almost complex structure.

the forms in \mathcal{A} are \mathfrak{g} -valued. Take the oriented orthonormal basis

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \quad (2.27)$$

The volume form for this orientation and metric is just $dx \wedge dy$, hence the Hodge dual gives us

$$J(dx) := *dx = dy \quad \text{and} \quad J(dy) = *dy = -dx \quad (2.28)$$

On the other hand the standard inner product on \mathbb{R}^2 induces a complex structure map (a counterclockwise quarter turn)

$$j \left(\frac{\partial}{\partial x} \right) = \frac{\partial}{\partial y} \quad \text{and} \quad j \left(\frac{\partial}{\partial y} \right) = -\frac{\partial}{\partial x} \quad (2.29)$$

The dual of j defines a linear operator J' on the space of 1-forms $A \in \mathcal{A}$

$$(J'(A)) \left(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \right) := A \left(j \left(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \right) \right) \quad (2.30)$$

where a and b are real coefficients. Using the above action of j a quick calculation shows

$$J'(dx) = -dy \quad \text{and} \quad J'(dy) = dx \quad (2.31)$$

which is clearly the opposite of J .

Hence if we complexify \mathbb{R}^2 then the holomorphic differential $dz_{J'}$ associated to J' is equal to the antiholomorphic differential $d\bar{z}_J$ associated to J . Let us complexify explicitly and produce the formulas for J' (then the reader

can check that the corresponding formulas for J are the conjugates). We have

$$\mathbb{R}^2 = \mathbb{R} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \quad (2.32)$$

Allowing complex coefficients gives

$$\mathbb{C}^2 = \mathbb{C} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \quad (2.33)$$

Define

$$\frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad (2.34)$$

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad (2.35)$$

Then it is easy to check (using the above formulas for J') that

$$J' \left(\frac{\partial}{\partial z} \right) = i \frac{\partial}{\partial z} \quad (2.36)$$

$$J' \left(\frac{\partial}{\partial \bar{z}} \right) = -i \frac{\partial}{\partial \bar{z}} \quad (2.37)$$

$$(2.38)$$

So the holomorphic tangent space (relative to J') is just

$$(T\mathbb{C}^2)^{(1,0)} := \mathbb{C} \left\{ \frac{\partial}{\partial z} \right\} \quad (2.39)$$

and the antiholomorphic tangent space (relative to J') is just

$$(T\mathbb{C}^2)^{(0,1)} := \mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}} \right\} \quad (2.40)$$

The same calculations end up conjugated when we use the complex structure J instead.

In view of this example we do not need a Riemannian structure on Σ in order to Kähler quantize, but merely a complex structure j . In the next section (using instead the conformal field theory approach) we dispense even with the complex structure.

2.3 Conformal field theory

In the last section we outlined Kähler quantization and described how to construct a finite-dimensional quantum Hilbert space \mathcal{H} associated to the initial time slice $\Sigma \times \{0\}$. In the Hamiltonian formalism (on the manifold $\Sigma \times I$) \mathcal{H} is evolved forward using the Hamiltonian H . However it is easy to verify that for Chern-Simons $H = 0$. There are no dynamics on $\Sigma \times I$ where (Σ, j) is a *closed* Riemann surface, hence we conclude that Kähler quantization is rather mundane. Furthermore the chiral WZW action appears on the boundary in Chern-Simons, but this was not used in Kähler quantization. Motivated by this we turn to the richer structure provided by conformal field theory (which agrees with Kähler quantization on closed Riemann surfaces Σ [BL94]).

A detailed analysis of the Wess-Zumino-Witten model is provided in (for example) [Koh02].¹⁶ However here we restrict ourselves to the axiomatic

¹⁶The strategy for the WZW model is to first *avoid* closed surfaces and instead study the WZW action on Riemann surfaces (Σ, j) with *at least* one boundary circle. The WZW action is not a priori well-defined on Riemann surfaces with boundary, however a study of the unit disk $(\Sigma, j) = D$ yields a construction based on a central extension of the loop group. Gluing laws can then be defined. In particular this defines the theory on *closed* Riemann surfaces since any such surface can be decomposed into two surfaces glued along nonempty boundary.

framework described in [Seg04]. The most primitive notion introduced by Segal is a **modular functor**.¹⁷ We mention that in the following we consider Riemann surfaces with labeled (colored) boundary circles. A boundary circle should be interpreted as the boundary of an excised disk containing a quasiparticle, and the color specifies the particle species. In addition we require that the boundary circles be *parameterized*. To make contact with our previous characterization of quasiparticles (and remain consistent with other treatments (see chapter 5 in [Tur94] and chapter 5 in [BK00]) it is not necessary to parameterize boundary circles, but rather merely select a *basepoint* on each boundary circle.¹⁸ A third alternative is to shrink each circle to a *marked point with distinguished tangent vector* on a *closed* surface Σ . These are **marked arcs**. However in CFT the boundary circles play a richer role - on the one hand they are quasiparticles, but on the other hand Riemann surfaces can be glued together along parameterized boundary circles (which cannot be done with marked arcs).

Definition 2.41. Let ϕ be a finite set of labels (particle species). Define a category \mathcal{G}_ϕ as follows:

1. An object is a compact Riemann surface (Σ, j) of arbitrary topological type, and possibly with many connected components and parameterized

¹⁷A modular functor is part of the underlying structure of a *chiral* conformal field theory (a *weak* conformal field theory in the language of [Seg04]). Given two opposite-chirality weak conformal field theories based on the same *unitary* modular functor it is possible to combine them to form an honest conformal field theory. Since it is a chiral theory that appears in Chern-Simons we restrict our attention to the modular functor.

¹⁸It is clear that a circle S^1 parameterized by a diffeomorphism $S^1 \rightarrow U(1)$ has a distinguished basepoint (e.g. the preimage of $\{1\}$ for example). However the space $\text{Diff}_{\text{pt}}^+(S^1)$ of all (orientation preserving) diffeomorphisms that share the same basepoint is contractible. Below we shall only be concerned with π_1 of the various spaces that appear, hence only the parameterization up to homotopy is important.

boundary circles. The boundary circles are labeled (**colored**) with elements from ϕ . If the orientation induced by the parameterization agrees with the boundary orientation then the circle is *outgoing*. If they disagree then the circle is *incoming*.

2. A morphism $(\Sigma, j) \rightarrow (\bar{\Sigma}, \bar{j})$ takes a Riemann surface (Σ, j) with an outgoing and an incoming boundary circle labeled by the same color $i \in \phi$ and glues them along the parameterizations to form a new Riemann surface $(\bar{\Sigma}, \bar{j})$ with two fewer boundary circles.

Definition 2.42. A **Segal modular functor** is a functor ¹⁹

$$\mathcal{F} : \mathcal{G}_\phi \rightarrow \text{finite dimensional complex vector spaces} \quad (2.43)$$

that assigns to a Riemann surface (Σ, j) with colored parameterized boundary a complex vector space $\mathcal{F}((\Sigma, j))$ (not a Hilbert space in general). This functor must satisfy

1. \mathcal{F} is a *holomorphic functor* (see below)
2. $\mathcal{F}((\Sigma, j) \amalg (\Sigma', j')) = \mathcal{F}((\Sigma, j)) \otimes \mathcal{F}((\Sigma', j'))$
3. For the Riemann sphere $\dim(\mathcal{F}(S^2)) = 1$

¹⁹We note that a Segal modular functor is stronger than the modular functor defined later in this treatment. A Segal modular functor is defined in terms of Riemann surfaces, boundary circles can be glued, and is valid in 2 dimensions only.

However, the dependence on the complex structure of a Riemann surface Σ can be relaxed. Presumably then a Segal modular functor is equivalent to an *extended 2-d modular functor* as discussed in chapter 5 of [Tur94] and chapter 5 of [BK00]. Because of the gluing property an extended 2-d modular functor is stronger than a modular functor defined below (and in chapter 3 of [Tur94]).

4. Consider cutting a Riemann surface (Σ, j) along a parameterized simple closed curve to produce a new surface with two more boundary circles (one incoming and one outgoing). Let us color both circles with a color i from the finite set of colors ϕ . Denote this new Riemann surface by (Σ_i, j) . We could think about sewing this back together, which by definition is just a morphism $f_i : (\Sigma_i, j) \rightarrow (\Sigma, j)$ (a gluing). The functor then gives a linear map $\mathcal{F}(f_i) : \mathcal{F}((\Sigma_i, j)) \rightarrow \mathcal{F}((\Sigma, j))$. Summing over all colors we require that the map

$$\bigoplus_{i \in \phi} \mathcal{F}((\Sigma_i, j)) \rightarrow \mathcal{F}((\Sigma, j)) \quad (2.44)$$

be a natural isomorphism.

In order to define holomorphic functor we mention some more standard results from complex geometry. Consider the space $\mathcal{J}(\Sigma)$ of *all* complex structures on Σ (Σ is a smooth manifold possibly with colored parameterized boundary). In other words $\mathcal{J}(\Sigma)$ is the space of all Riemann surfaces that are topologically diffeomorphic to Σ . $\mathcal{J}(\Sigma)$ is a contractible topological space (consider the space of smoothly-varying matrices $j(x)$ for $x \in \Sigma$ such that $j^2 = -1$. This space is contractible in 2 dimensions).

Two Riemann surfaces (Σ, j_1) and (Σ, j_2) of the same topological type are *equivalent* if there is an orientation-preserving diffeomorphism $\phi : \Sigma \rightarrow \Sigma$ that maps j_1 to j_2 (i.e. a biholomorphic map). If Σ has boundary then we assume that ϕ maps circles to circles respecting the parameterizations. The resulting space $\mathcal{J}(\Sigma)/\sim$ is the *moduli space* \mathcal{C}_Σ (see [Sch89] - except note that in contrast to other treatments here any boundary components are parameterized).

A functor \mathcal{F} is **holomorphic** if the complex vector spaces $\mathcal{F}((\Sigma, j))$ assigned to Riemann surfaces (Σ, j) of a given topological type Σ smoothly vary as the complex structure j varies. More precisely, \mathcal{F} is holomorphic if we obtain a holomorphic vector bundle $\mathcal{F}(\mathcal{C}_\Sigma) \rightarrow \mathcal{C}_\Sigma$.

Consider the following (proposition 5.4 in [Seg04]): ²⁰

Proposition 2.45 (Segal). *Associated to any arbitrary modular functor \mathcal{F} is a canonical flat connection on the **projective** bundle $\mathbb{P}\mathcal{F}(\mathcal{C}_\Sigma) \rightarrow \mathcal{C}_\Sigma$*

This implies that we can identify the projective vector spaces $\mathbb{P}\mathcal{F}((\Sigma, j_1))$ and $\mathbb{P}\mathcal{F}((\Sigma, j_2))$ once a path has been specified in \mathcal{C}_Σ from (Σ, j_1) to (Σ, j_2) . Since the connection is *flat* only the homotopy type of the path is relevant.

Choose a complex structure (Σ, j) and associate to Σ the vector space

$$\mathcal{H} := \mathcal{F}((\Sigma, j)) \tag{2.46}$$

From the comments above \mathcal{H} is a projective representation of $\pi_1(\mathcal{C}_\Sigma)$, and the choice of j is equivalent to the choice of basepoint for $\pi_1(\mathcal{C}_\Sigma)$. Let us now study $\pi_1(\mathcal{C}_\Sigma)$.

Example 2.47. It is a standard result that when Σ is a *closed* oriented surface then \mathcal{C}_Σ is a finite-dimensional complex variety but perhaps *with singularities*.

For the Riemann sphere the moduli space \mathcal{C}_{S^2} is a point (there is a unique Riemann sphere $\mathbb{C} \cup \{\infty\}$ up to automorphisms of the complex structure via the action of $\mathrm{PSL}(2, \mathbb{C})$).

A closed genus 1 surface is obtained from the complex plane \mathbb{C} in the usual way by identifying points related by translations using a rank 2 lattice

²⁰Unfortunately Segal avoids proving this for *closed* oriented surfaces Σ since then the moduli space \mathcal{C}_Σ may have singularities. We ignore this source of complication.

$\mathbb{Z} \oplus \mathbb{Z}$. Explicitly we identify $z \mapsto z + 1$ and $z \mapsto z + \tau$ where $\text{Im}(\tau) > 0$.²¹ Hence the complex tori are determined by a choice of $\tau \in U$ in the upper half plane.

However given a fixed lattice in \mathbb{C} even a basis of the form $(\tau, 1)$ is not unique. We can apply a unimodular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ ²² to the basis $(\tau, 1)$ to give a new basis $(a\tau + b, c\tau + d)$ for the same lattice. Next let us again use the automorphisms of the complex plane (affine transformations) to put this new basis back into the form $(\tau', 1)$. A small calculation shows that

$$\tau' = \frac{a\tau + b}{c\tau + d} \tag{2.48}$$

We note that we can multiply both numerator and denominator in the above equation by -1 and still get the same τ' , hence we need only consider projective unimodular matrices $\text{PSL}(2, \mathbb{Z})$. Summarizing, two complex tori are equivalent if related by a transformation in $\text{PSL}(2, \mathbb{Z})$ acting on the upper half plane U . Since $\text{PSL}(2, \mathbb{Z})$ is discrete we have that the action on U is *discontinuous* in the sense of [FK92] pg. 203. It can be shown that *as a naive set*

$$\mathcal{C}_T \cong U/\text{PSL}(2, \mathbb{Z}) = \mathbb{C} \tag{2.49}$$

However, we note that the above action is *not free*. Hence \mathcal{C}_T is not a smooth manifold, but in fact has 2 singular points with extra internal structure. In other words the moduli space \mathcal{C}_T is a *stack* and it is *not true* that

²¹We see that the universal covering space of a torus is just \mathbb{C} . The automorphisms (transformations that preserve the complex structure) of \mathbb{C} are just the affine transformations $z \mapsto az + b$ where $a, b \in \mathbb{C}$ and $a \neq 0$. Using these automorphisms we can transform a given lattice generated by arbitrary vectors α and β into a unique lattice generated by vectors of the form 1 and τ with $\text{Im}(\tau) > 0$. The resulting complex structure on the torus is unaffected.

²²Explicitly $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$

$\pi_1(\mathcal{C}_T) = \pi_1(\mathbb{C}) = 1$. In fact it turns out (for a suitably-defined definition of the fundamental group) that $\pi_1^{\text{stack}}(\mathcal{C}_T) \cong \text{MCG}(T) \cong \text{PSp}(2, \mathbb{Z}) \cong \text{PSL}(2, \mathbb{Z})$ where $\text{MCG}(T)$ is the mapping class group of the torus.

For closed higher genus (≥ 2) surfaces a similar result holds (technically the construction is easier because a *fine* moduli space can be extracted from the coarse moduli space). It happens that $\dim_{\mathbb{C}} \mathcal{C}_{\Sigma} = 3g - 3$, but again there are singularities which force us to treat \mathcal{C}_{Σ} as a stack. It turns out that again

$$\pi_1^{\text{stack}}(\mathcal{C}_{\Sigma}) \cong \text{MCG}(\Sigma) \tag{2.50}$$

We refer the reader to chapter 6 of [BK00].

Example 2.51. Now let us consider surfaces with *parameterized* holes.²³ In this case there are no singularities in \mathcal{C}_{Σ} .²⁴ The k -holed sphere requires special treatment and must be dealt with separately in the three regimes $k = 1$, $k = 2$, and $k \geq 3$. The k -holed torus also must be analyzed by hand in the regimes $k = 1$ and $k \geq 2$. Higher genus ($g \geq 2$) surfaces can be dealt with uniformly, although much is still unknown.²⁵ We start with the sphere.

First, let us consider the sphere with one parameterized hole, i.e. the unit disk $\Delta = \{z : |z| \leq 1\}$.²⁶ The unit disk conformally maps to the upper half plane U via the map $z \mapsto i\frac{1-z}{1+z}$, and the upper half plane has a

²³from now on by “hole” we mean a removed open disk, i.e. Σ has parameterized boundary circles. This is in contrast to a puncture, i.e. a removed point - see [FM07] pg. 64

²⁴we emphasize that the boundary here is parameterized. For a contrasting example suppose Σ is an annulus with *unparameterized boundary*. Then the moduli space is the real interval $(0, 1)$, which disagrees with the result stated here. See [FK92] page 211

²⁵see [Bir74], although here we have the additional complication of parameterized holes rather than simple punctures

²⁶By the classification of *exceptional* Riemann surfaces the only simply connected Riemann surfaces are $\mathbb{C} \cup \{\infty\}$, \mathbb{C} , and $\Delta = \{z : |z| \leq 1\}$. Hence there is only one “disk” to consider here. See [FK92] pg. 207

unique complex structure, hence there is a unique complex structure on the unit disk Δ . So we expect that $\mathcal{C}_\Delta \cong \{\text{pt}\}$. However, we have forgotten about the parameterization of the boundary S^1 so we must take into account the group $\text{Diff}^+(S^1)$. To make the analysis easier for our purposes it suffices to think about the boundary with a distinguished basepoint (rather than a full parameterization). Hence let us consider the upper half plane U with a distinguished basepoint on the real axis.

The automorphism group (the group that preserves the complex structure) of the upper half plane U is just $PSL(2, \mathbb{R})$. In particular we can think about the affine transformation $z \mapsto z + a$ for any real number a . But this maps any choice of basepoint on the real axis to any other choice of basepoint, so we conclude that the choice of basepoint is irrelevant.²⁷ Hence even with a parameterized boundary we have $\mathcal{C}_\Delta \cong \{\text{pt}\}$.

Example 2.52. Now consider a sphere with two parameterized holes (an annulus). Again by the classification for *exceptional* Riemann surfaces the only Riemann surfaces with $\pi_1(\Sigma) \cong \mathbb{Z}$ are $\mathbb{C} \setminus \{0\}$, $\Delta \setminus \{0\}$, and the family of standard annuli $\Delta_r = \{z \in \mathbb{C} : r \leq |z| \leq 1\}$ where $r \in (0, 1)$ (i.e. *all* annuli are just standard annuli). Hence (as we have already mentioned) the moduli space of complex annuli \mathcal{C}_{Δ_r} is just the interval $(0, 1)$. Here again, however, we have forgotten the boundary parameterizations. Like before (and from now on) we do not consider the full parameterizations, but rather a distinguished basepoint on each boundary circle. It is clear that we can perform a rigid rotation (which preserves the complex structure on the annulus) to rotate any

²⁷More trivially instead we could just think of rigid rotations acting on the unit disk Δ (these preserve the complex structure). Any arbitrary basepoint on the boundary circle can be rotated to the point $z = 1$.

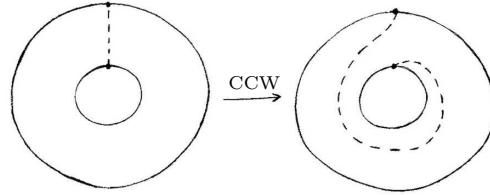


Figure 2.1: A counterclockwise twist of a boundary circle with respect to a second boundary circle. Instead of using parameterizations we depict distinguished basepoints. We provide visual markings to show the diffeomorphism.

arbitrary basepoint on the outer circle $\{z \in \mathbb{C} : |z| = 1\}$ to the point $z = 1$, hence the choice of basepoint on the outer circle is irrelevant.

Now we have used up the rigid rotation automorphism (which is the only automorphism of an annulus) hence we cannot dispense with the choice of basepoint on the inner circle (we have a whole S^1 worth of choices). In view of this we see that the moduli space of annuli (with parameterized boundary) is just $\mathcal{C}_{\Delta_r} \cong (0, 1) \times S^1$.²⁸ There are no singularities nor stack structure, hence we directly calculate $\pi_1(\mathcal{C}_{\Delta_r}) \cong \mathbb{Z} \cong \text{MCG}_{\partial}(\Delta_r)$.²⁹ See figure (2.1)

Example 2.53. More generally recall that we saw for closed surfaces $\pi_1^{\text{stack}}(\mathcal{C}_{\Sigma}) \cong \text{MCG}(\Sigma)$. For compact oriented surfaces with $k \geq 1$ holes we now sketch that the same result is true although the presence of parameterized boundary circles enlarges the mapping class group considerably (for a detailed account see [Bir74] and [FM07]).

In order to understand $\pi_1(\mathcal{C}_{\Sigma})$ let us remind ourselves that previously we obtained the moduli space \mathcal{C}_{Σ} from $\mathcal{J}(\Sigma)$ by identifying any two Riemann

²⁸This is merely a homotopy equivalence because we are considering basepoints rather than parameterizations.

²⁹This is an enlarged mapping class group for surfaces with *basepointed* boundary circles. In this case a Dehn twist in a collar neighborhood of a boundary circle is a non-trivial element of the mapping class group. If the boundary circles were not parameterized/basepointed then such a Dehn twist could be smoothly deformed (untwisted) back to the identity.

surfaces (Σ, j_1) and (Σ, j_2) if there is a biholomorphic diffeomorphism $\Sigma \rightarrow \Sigma$ mapping one complex structure to the other. Now we have boundary circles (equipped with basepoints) hence we further require that any diffeomorphism maps basepoints to basepoints. Denote this space of biholomorphic basepoint-preserving diffeomorphisms $\text{Diff}_\partial^+(\Sigma)$.³⁰

Rather than mod out by *all* such biholomorphic diffeomorphisms let us consider a weaker notion of equivalence by defining **Teichmüller space** \mathcal{T}_Σ where we identify any two Riemann surfaces if there is a biholomorphic diffeomorphism in $\text{Diff}_\partial^+(\Sigma)$ *that can be smoothly deformed to the identity* (clearly such diffeomorphisms must be the identity on each boundary circle separately). Denote this restricted subset $\text{Diff}_{\partial,0}^+(\Sigma) \subset \text{Diff}_\partial^+(\Sigma)$. In symbols we have

$$\mathcal{C}_\Sigma := \mathcal{J}(\Sigma)/\text{Diff}_\partial^+(\Sigma) \tag{2.54}$$

and

$$\mathcal{T}_\Sigma := \mathcal{J}(\Sigma)/\text{Diff}_{\partial,0}^+(\Sigma) \tag{2.55}$$

On the other hand by definition $\text{Diff}_\partial^+(\Sigma)/\text{Diff}_{\partial,0}^+(\Sigma)$ is the mapping class group $\text{MCG}_\partial(\Sigma)$, so we see that

$$\mathcal{C}_\Sigma = \mathcal{T}_\Sigma/\text{MCG}_\partial(\Sigma) \tag{2.56}$$

In this context the mapping class group is often called the **Teichmüller group** $\text{Teich}(\Sigma)$.

In the language of covering space theory we can view Teichmüller space

³⁰obviously the diffeomorphism must preserve orientation as well.

as a covering of moduli space

$$\mathcal{T}_\Sigma \rightarrow \mathcal{C}_\Sigma \tag{2.57}$$

where the deck transformations are just given by elements of $\text{MCG}_\partial(\Sigma)$. The usual covering space results tell us that ³¹

$$\text{Deck Transformations} \cong \pi_1(\mathcal{C}_\Sigma)/\pi_1(\mathcal{T}_\Sigma) \tag{2.58}$$

$$\text{MCG}_\partial(\Sigma) \cong \pi_1(\mathcal{C}_\Sigma)/1 \tag{2.59}$$

In the above equation we have used the fact that $\mathcal{J}(\Sigma)$ is actually a contractible space, and since modding out by diffeomorphisms that can be deformed to the identity does not change the homotopy type, we see that the Teichmüller space \mathcal{T}_Σ is also contractible. So $\pi_1(\mathcal{T}_\Sigma) = 1$. ³²

Example 2.60. We have shown that for arbitrary compact oriented surfaces with/without parameterized holes that

$$\pi_1(\mathcal{C}_\Sigma) \cong \text{MCG}_\partial(\Sigma) \tag{2.61}$$

for suitably defined fundamental group and mapping class group. Hence it is worthwhile to study $\text{MCG}_\partial(\Sigma)$ a bit further. We already mentioned the explicit results for the sphere with $k = 0$, $k = 1$, and $k = 2$ punctures. We also mentioned that for the closed torus $\text{MCG}_\partial(T) \cong \text{PSL}(2, \mathbb{Z})$.

Now consider a special family of examples - the *unit disk* with $k \geq 2$ parameterized holes in the interior. This is *not* the sphere with $k + 1$ holes

³¹We can use this result if the deck group action is free, which is evidenced by the fact that the resulting quotient manifold \mathcal{C}_Σ has no singularities.

³²This explains the somewhat interchangeable roles that $\pi_1(\mathcal{C}_\Sigma)$, $\text{MCG}_\partial(\Sigma)$, and $\text{Teich}(\Sigma)$ play in the literature.

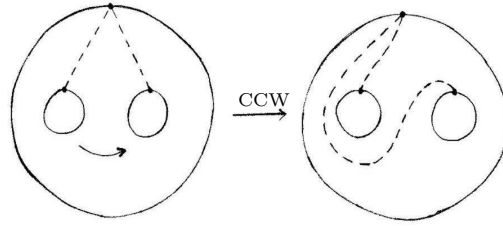


Figure 2.2: A counterclockwise braiding of two boundary circles with respect to the distinguished outer boundary circle. Instead of using parameterizations we depict distinguished basepoints. We provide visual markings to show the diffeomorphism.

because here the outer $(k+1)$ st boundary circle is considered *distinguished and fixed*. These disks can be used as building blocks to analyze certain aspects of all surfaces.

For concreteness consider the two-holed disk ($k = 2$) Δ_2 embedded in \mathbb{R}^2 using whatever standard embedding that the reader prefers (see the left disk in figure (2.2) for our convention).

Now consider the counterclockwise *braiding* diffeomorphism $c : \Delta_2 \rightarrow \Delta_2$ depicted in figure (2.2). This is a diffeomorphism of Δ_2 that cannot be smoothly deformed to the identity, hence is a nontrivial element of the mapping class group. More generally for a disk with k parameterized holes we expect that the *braid group* B_k on k strands is a subgroup of $\text{MCG}_\partial(\Sigma)$.

Likewise each of the interior holes can be (separately) *twisted* via a full counterclockwise Dehn twist $\theta_i : \Delta_2 \rightarrow \Delta_2$ for $i = 1, \dots, k$ (see figure (2.1) for the case $k = 1$). Hence we convince ourselves that \mathbb{Z}^k is a subgroup of $\text{MCG}_\partial(\Sigma)$.

It is not difficult to see that a braiding operation, followed by any twist operation, followed by the inverse braiding operation, can be written as a different twist operation. In other words B_k is in the normalizer for \mathbb{Z}^k .

In light of this it is not surprising that $\text{MCG}_\partial(\Sigma)$ is the semidirect product of \mathbb{Z}^k with B_k :

$$\text{MCG}_\partial(\Sigma) \cong \mathbb{Z}^k \rtimes B_k \tag{2.62}$$

We have been incomplete in our analysis, however, since we have forgotten that in conformal field theory each boundary component must be labelled by a *color* from a finite set. It only makes sense to swap holes that have the same coloring, so we are forced to consider instead of the full braid group B_k the *colored braid group* CB_k . So we have

$$\text{MCG}_\partial(\Sigma) \cong \mathbb{Z}^k \rtimes CB_k \tag{2.63}$$

Example 2.64. Now consider the special case of the sphere with k parameterized holes. It is fairly trivial to analyze this case by excising a *special disk* (from the last example) that contains all of the holes. The result is two pieces - a disk Δ and a disk Δ_k with k holes. Then the mapping class group is

$$\text{MCG}_\partial(\Sigma) \cong (\mathbb{Z}^k \rtimes CB_k) / \mathbb{Z}_{\text{everything}} \tag{2.65}$$

$\mathbb{Z}_{\text{everything}}$ is the subgroup of $\mathbb{Z}^k \rtimes CB_k$ generated by the central element that takes a full Dehn twist of the entire interior of Δ_k (leaving the outer circle fixed, of course). When the disks are glued together this Dehn twist can be pushed onto Δ instead, and any Dehn twist of Δ can be smoothly deformed to the identity. So we conclude that $\mathbb{Z}_{\text{everything}}$ is trivial for the sphere with holes.

For example, for $k = 2$ holes (with the same coloring) the braid group B_2 becomes the symmetric group S_2 when modding out by $\mathbb{Z}_{\text{everything}}$.

$\text{MCG}(\Sigma)$ in genus $g \geq 1$ is significantly more complicated and much is not known. We refer the reader to [FM07].

2.4 Axiomatic definition of an $(n+1)$ -dimensional TQFT

The axioms for an $(n + 1)$ -dimensional TQFT were originally proposed by Atiyah (see, e.g., [Ati90a]). They appear in various incarnations throughout the literature, but we follow chapter 3 of [Tur94].

Modular functor

Consider the category \mathcal{U} defined by

1. The objects are (possibly *extended*) n -dimensional closed oriented manifolds Σ . We are interested in the case $n = 2$, and for us the extended structure on a closed genus g surface Σ is a *parameterization* diffeomorphism

$$\phi : \Sigma_g^{\text{standard}} \rightarrow \Sigma \tag{2.66}$$

where $\Sigma_g^{\text{standard}}$ is a fixed genus g surface. ³³

2. The morphisms are orientation-preserving diffeomorphisms $\Sigma \rightarrow \Sigma'$.

\mathcal{U} has a canonical commutative strict monoidal structure (see chapter (4)):

³³The parameterization can be relaxed to a much weaker extended structure. See [Ati90b],[Wal91],[FG91].

1. The tensor product is given by disjoint union:

$$\Sigma \boxtimes \Sigma' := \Sigma \sqcup \Sigma' \quad (2.67)$$

2. The unit object $\mathbb{1}$ is the empty set \emptyset (since $\Sigma \sqcup \emptyset = \Sigma$).
3. \mathcal{U} is commutative, i.e. $\Sigma \sqcup \Sigma' = \Sigma' \sqcup \Sigma$.

Now consider the category $\text{Vect}_{\mathbb{C}}^{\text{fin}}$ of finite-dimensional complex vector spaces. This is also a commutative strict monoidal category (using the ordinary vector space tensor product \otimes). The unit object here is \mathbb{C} .

Definition 2.68. A **modular functor** \mathcal{F} is a covariant strict monoidal functor (see chapter (5))

$$\mathcal{F} : \mathcal{U} \rightarrow \text{Vect}_{\mathbb{C}}^{\text{fin}} \quad (2.69)$$

In other words, to each n -dimensional extended closed oriented manifold Σ we assign a vector space $\mathcal{F}(\Sigma)$:

$$\Sigma \xrightarrow{\mathcal{F}} \mathcal{F}(\Sigma) \quad (2.70)$$

To each orientation-preserving diffeomorphism $f : \Sigma \rightarrow \Sigma'$ we assign a vector space isomorphism $\mathcal{F}(f) : \mathcal{F}(\Sigma) \rightarrow \mathcal{F}(\Sigma')$ (which we denote $f_{\#}$):

$$f \xrightarrow{\mathcal{F}} f_{\#} \quad (2.71)$$

Functoriality means $(fg)_{\#} = f_{\#}g_{\#}$ and $\text{id}_{\Sigma} \mapsto \text{id}_{\#} = \text{id}_{\mathcal{F}(\Sigma)}$.

Being a strict monoidal functor means that in addition

$$\mathcal{F}(\Sigma \sqcup \Sigma') = \mathcal{F}(\Sigma) \otimes \mathcal{F}(\Sigma') \quad (2.72)$$

There are extra associativity and naturality axioms for strict monoidal functors that can be found in chapter (5). Most notably we have the identity assignment

$$\mathcal{F}(\emptyset) = \mathbb{C} \tag{2.73}$$

It is interesting to contrast with the Segal modular functor in section 2.3. Most conspicuous is the lack of gluing in this version. An $n = 2$ modular functor as defined here is weaker than a Segal modular functor.³⁴ We mention that the extended structure on Σ for the case $n = 2$ can be weakened to a choice of distinguished Lagrangian subspace of $H_1(\Sigma)$.

$(n + 1)$ -dimensional TQFT

We require 2 more categories. First consider the bordism category Bord_{n+1} defined by

1. The objects are the same as the objects in \mathcal{U} (extended closed oriented n -manifolds).
2. The morphisms are $(n + 1)$ -dimensional compact oriented bordisms, i.e. for objects Σ and Σ' a morphism $\Sigma \rightarrow \Sigma'$ is an $(n + 1)$ -dimensional oriented manifold X such that $\partial X = -\Sigma \sqcup \Sigma'$. The bordisms may also have extended structure.³⁵

³⁴The nomenclature is confusing. In chapter 5 of [Tur94] is described a so-called 2- d modular functor. The construction has much more structure than a modular functor in 2 dimensions (as defined here and in chapter 3 of [Tur94]). Following [BK00] we prefer to call the stronger version an **extended 2-d modular functor**. Presumably extended 2-d modular functors are in one-to-one correspondence with the Segal modular functors defined above.

³⁵For a $(2 + 1)$ -dimensional theory there is no need to endow bordisms with extended structure in order to define a theory with anomaly (see below). However an anomaly-free

Consider a different category of bordisms \mathcal{B} defined by

1. The objects X in \mathcal{B} are the morphisms in Bord_{n+1} , i.e. (extended) compact oriented $(n+1)$ -dimensional *bordisms* between extended oriented closed n -manifolds.
2. The morphisms are orientation-preserving diffeomorphisms between bordisms $f : X \rightarrow X'$.

\mathcal{B} has a canonical commutative strict monoidal structure:

1. The tensor product is given by disjoint union:

$$X \boxtimes X' := X \sqcup X' \tag{2.74}$$

2. The unit object $\mathbb{1}$ is the empty set \emptyset (since $X \sqcup \emptyset = X$).
3. \mathcal{B} is commutative, i.e. $X \sqcup X' = X' \sqcup X$.

Definition 2.75. An $(n+1)$ -dimensional topological quantum field theory τ based on $(\mathcal{F}, \mathcal{U}, \text{Bord}_{n+1}, \mathcal{B})$ is a rule:

1. Given a bordism $X \in \text{Mor}(\Sigma, \Sigma')$ between $\Sigma \in \text{Ob}(\text{Bord}_{n+1})$ and $\Sigma' \in \text{Ob}(\text{Bord}_{n+1})$ assign a linear map

$$\tau(X) : \mathcal{F}(\Sigma) \rightarrow \mathcal{F}(\Sigma') \tag{2.76}$$

2. This rule must be *projectively functorial* with respect to the category Bord_{n+1} (i.e. satisfy a gluing property). Consider a bordism X between

theory requires an extended structure on X (in the language of [Tur94] these are *weighted extended bordisms*). See [Ati90b],[Wal91],[FG91].

Σ and Σ' and another bordism X' between Σ' and Σ'' . Then glue the bordisms together along Σ' to form a bordism $X \cup_{\text{glue}} X' : \Sigma \rightarrow \Sigma''$. We require that: ³⁶

$$\tau(X \cup_{\text{glue}} X') = k\tau(X') \circ \tau(X) \quad (2.77)$$

where $k \in \mathbb{C}^\times$ is an invertible number called the **gluing anomaly** (if $k = 1$ then the theory is said to be *anomaly-free*).

Since the cylinder $\Sigma \times I$ is the identity morphism $\Sigma \rightarrow \Sigma$ in the category Bord_{n+1} , *projective functoriality* also requires that

$$\tau(\Sigma \times I) = \text{id}_{\mathcal{F}(\Sigma)} \quad (2.78)$$

3. In terms of the category \mathcal{B} we have an assignment

$$\tau : \mathcal{B} \rightarrow \text{finite-dim linear maps} \quad (2.79)$$

We require this map be a strict monoidal functor. This means (among other things) that

$$\tau(X_1 \sqcup X_2) = \tau(X_1) \otimes \tau(X_2) \quad (2.80)$$

4. Finally we require a compatibility on the categories \mathcal{U} , \mathcal{B} , and Bord_{n+1} : if $f : X \rightarrow X'$ is a morphism in \mathcal{B} ($f : X \rightarrow X'$ is an orientation-preserving diffeomorphism of bordisms) then the following diagram must

³⁶The anomaly k measures how far τ is from being a functor $\text{Bord}_{n+1} \rightarrow \text{Vect}_{\mathbb{C}}^{\text{fin}}$.

commute:

$$\begin{array}{ccc}
 \mathcal{F}(\partial_- X) & \xrightarrow{\tau(X)} & \mathcal{F}(\partial_+ X) \\
 \downarrow (f|_{\partial_- X})_{\#} & & \downarrow (f|_{\partial_+ X})_{\#} \\
 \mathcal{F}(\partial_- X') & \xrightarrow{\tau(X')} & \mathcal{F}(\partial_+ X')
 \end{array} \tag{2.81}$$

Extended $(2 + 1)$ -dim TQFTs and extended 2-d modular functors

The definition of TQFT provided above applies in any dimension. However in $(2 + 1)$ -dimensions most known theories satisfy stronger properties and can be interpreted as **extended $(2 + 1)$ -dim TQFT** (or *TQFT with corners*). We refer the reader to chapter 4 of [BK00] for the relevant extended axioms,³⁷ but briefly this means that the objects in \mathcal{U} are not *closed* 2-surfaces, but instead are compact surfaces with marked arcs (or parameterized boundary circles). The bordisms are also extended to include colored ribbon graphs with ends that terminate on the marked arcs. The construction provided here in chapter (4) is manifestly *extended*.

Likewise, the notion of modular functor can be strengthened to an **extended 2-d modular functor** (see chapter 5 of [BK00]). The main additional feature is that colored boundary circles are allowed, and they can be glued (compare with the Segal modular functor).

The known causality relationships between these notions are depicted

³⁷In particular the theories of Deloup described in [Del99],[Del01],[Del03] are *not* extended. Links are intrinsic in the construction, however ribbon graphs do not appear. Furthermore the boundary surfaces are always *closed* manifolds.

in the following diagram (as described in section 5.8 of [BK00]):

$$\begin{array}{ccccc}
 \text{Modular} & & \text{Extended} & & \\
 \text{Tensor} & \xRightarrow{\quad} & (2+1)\text{-dim TQFT} & \xRightarrow{\quad} & (2+1)\text{-dim TQFT} \\
 \text{Category} & \xleftarrow{\quad} & & \xrightarrow{\quad} & \\
 & & \Downarrow & & \Downarrow \\
 & & \text{Extended 2-d Modular Functor} & \xRightarrow{\quad} & \text{2-d Modular Functor} \\
 & & \uparrow \text{?} & & \\
 & & \text{Segal Modular Functor} & &
 \end{array}
 \tag{2.82}$$

The broken line indicates that under certain circumstances an extended 2-d modular functor reproduces a modular tensor category (see theorem 5.7.10 in [BK00]).

Chapter 3

Toral Chern-Simons Theories

In this chapter we aim to give a brief summary of toral Chern-Simons theories as described by Belov and Moore in [BM05]. Belov and Moore give a much more general description that includes spin TQFTs, but in the context of modular tensor categories we are confined to ordinary TQFTs. Hence in this paper we shall mostly limit ourselves to the ordinary (non-spin) Chern-Simons theories.

We will strive to keep the notation found in [BM05] to avoid confusion.

3.1 Classical toral Chern-Simons theories

Classical Chern-Simons theories for arbitrary compact gauge groups were studied by Freed in [Fre95, Fre], and the $U(1)$ theory was studied later by Manoliu [Man98].

To begin we consider Chern-Simons theory for an arbitrary gauge group G . Let X_3 be a compact oriented 3-manifold. Let $P \rightarrow X_3$ be a principal

G -bundle equipped with a connection Θ .¹ let ρ be a real representation of G , and let $E \rightarrow X_3$ be the associated vector bundle. Denote by D the covariant derivative induced on E by the connection Θ on P . If we assume that P is trivializable² and that a global section $p : X_3 \rightarrow P$ has been chosen then the trivialization induces the *standard flat connection* Θ^0 . Since P is trivializable we know that necessarily the associated bundle E is trivializable.³ In that case the standard flat Ehresmann connection Θ^0 induces a standard flat covariant derivative D^0 on E , and since the two covariant derivatives live in the same affine space we can write $D = D^0 + A$ for some vector potential $A \in \Omega^1(X_3; \text{End}(E))$. In a local trivialization of E near a given point $x \in X_3$ we can think of the vector potential as a 1-form on X_3 valued in (a subgroup of) $\mathfrak{gl}(n)$ where n is the dimension of E .

However D and D_0 are not merely arbitrary $\mathfrak{gl}(n)$ connections, but rather they arise from connections on a principal G -bundle (they are G -connections). Hence they respect the underlying G -structure of the vector bundle E . For example, if $G = SO(n)$ then E comes equipped with an inner product. In that case we can imagine picking an orthonormal basis in a fiber and parallel transport it to neighboring fibers. The result should be an orthonormal basis in the other fibers as well. This is in contrast to a general $\mathfrak{gl}(n)$ connection that may evolve the orthonormal basis into an arbitrary basis in the other fibers.

So we require that the vector potential is actually Lie algebra valued, i.e. $A \in \Omega^1(X_3; \mathfrak{g})$. In the $G = SO(n)$ example the vector potential is valued

¹a G -equivariant \mathfrak{g} -valued horizontal 1-form on P , i.e. an Ehresmann connection.

² P admits a global section.

³For a straightforward account of Chern-Simons actions for trivializable bundles see [BM94].

in $\mathfrak{so}(n)$ (infinitesimal rigid rotations).

Since the vector potential A is endomorphism-valued (actually Lie algebra-valued), and in any fiber an endomorphism is just a matrix (with respect to an arbitrary basis), there is a well-defined trace Tr .⁴ Therefore the Chern-Simons action integral

$$\int_{X_3} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \quad (3.1)$$

is well-defined (we multiply matrices where necessary before taking the trace).

In order to proceed further it is necessary to consider the role of the choice of trivialization on E that defines D^0 . Recall that a gauge transformation is a smooth map $X_3 \rightarrow G$. Given a point $x \in X_3$ the group element $g(x) \in G$ acts on the fiber P_x (or acts on the vector bundle fiber E_x). In particular a gauge transformation maps the chosen global section (trivialization) to a new global section (trivialization). Conversely, it is clear that *any two* trivializations are related by such a gauge transformation. Hence different choices of trivialization can be neatly encoded in the group of gauge transformations.

On the other hand it is a basic physical axiom of gauge theory that if two connections are related by a gauge transformation then they are physically indistinguishable, i.e. the mathematical description of a gauge theory is redundant. After modding out by gauge transformations we see that the choice of trivialization is irrelevant. Hence it is enough for P to be *trivializable*.⁵

In order for this physical axiom to be consistent, however, it is necessary to check that gauge transformations do not alter the action integral above. It

⁴which is independent of the basis by cyclicity.

⁵Let us clarify that choosing a trivialization is *not* gauge fixing. We have merely argued that the choice of trivialization is irrelevant up to gauge. Even armed with a choice of trivialization we must still mod out by the group of gauge transformations. The gauge axiom actually shrinks configuration space (which has the pleasant side-effect that the choice of trivialization is inconsequential).

is straightforward to check that this integral is invariant under small gauge transformations (gauge transformations that can be continuously deformed to the trivial gauge transformation).

Under large gauge transformations, however, the integral shifts by an integer N (if A is properly normalized) and hence is not well defined. However the complex exponential

$$\exp\left(2\pi i k \int_{X_3} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)\right) \quad (3.2)$$

is well-defined, even under large gauge transformations, for any integer k (the *level*).

The above construction relied on the assumption that E is trivialized. It is a fact that if G is a connected and simply-connected Lie group (e.g. $SU(2)$) then (see for example [DW90]) any principal G -bundle over a manifold of dimension ≤ 3 is trivializable. In that case this construction is always well-defined.

On the other hand, $U(1)^N$ is not simply-connected and it is *not* true that any principal $U(1)^N$ -bundle over a 3-manifold is trivializable. Nevertheless, for now we write explicitly the above action for $U(1)$ where we *assume* that E is trivialized:

$$\exp\left(2\pi i k \int_{X_3} \text{Tr}(A \wedge dA)\right) \quad (3.3)$$

Any *real* irreducible representation of $U(1)$ is a two-dimensional vector space where an element $e^{i\theta}$ acts by a rigid rotation

$$\begin{pmatrix} \cos(2\pi n\theta) & \sin(2\pi n\theta) \\ -\sin(2\pi n\theta) & \cos(2\pi n\theta) \end{pmatrix} \quad (3.4)$$

For each integer n there is a unique irreducible representation ρ_n . The Lie algebra is $\mathfrak{u}(1) \cong i\mathbb{R}$.

Hence we can dispense with trace Tr and the representation ρ_n and simply work in the defining representation $n = 1$ at the cost of banishing an integer to the front of the action integral:

$$\exp\left(2\pi i k n^2 \int_{X_3} (A \wedge dA)\right) \quad (3.5)$$

Here it is understood that A is now valued in the defining representation.

More generally any (not necessarily irreducible) real representation is written as a direct sum of irreducibles $\bigoplus m_n \rho_n$ where m_n is the multiplicity (only nonzero for finitely-many ρ_n 's). For such representations the vector potential A is no longer valued in $i\mathbb{R}$, but rather it is valued in diagonal matrices with values in $i\mathbb{R}$ along the diagonal. Taking the trace gives $(m_{n_1} n_1^2 + m_{n_2} n_2^2 + \dots) A \wedge dA$ where now once again A takes values in the defining representation.

We can absorb this arbitrary integer into the level k . Then we have the $U(1)$ Chern-Simons action for a trivialized bundle

$$\exp\left(2\pi i k \int_{X_3} A \wedge dA\right) \quad (3.6)$$

Notice that we no longer need to specify the representation because A is assumed to be in the defining representation ρ_1 .

For $U(1)$ theories it is much more convenient to define $B = 2k$ to be the “level”. B is obviously an *even* integer for the ordinary theories that we are describing now:

$$\exp\left(\pi i B \int_{X_3} A \wedge dA\right) \quad (3.7)$$

If the bundle E is not trivializable, however, then a different technique must be used to define the Chern-Simons action [DW90]. Choose a 4-manifold Z_4 such that X_3 is the boundary of Z_4 (such a manifold always exists by Rokhlin's theorem [PS96, pg. 87]).⁶ In some cases (depending on the gauge group G) the bundle E can be extended to a G -bundle $\tilde{E} \rightarrow Z_4$. For G a torus this is always possible.⁷

Next arbitrarily extend the connection D on E to a connection \tilde{D} on \tilde{E} (using a partition of unity). If \tilde{F} denotes the curvature of \tilde{D} then we can define the Chern-Simons action to be the integral of the second Chern class

$$\exp\left(\pi i B \int_{Z_4} \text{Tr}(\tilde{F} \wedge \tilde{F})\right) \quad (3.8)$$

We have used similar arguments as those above to absorb the representation information into the even-valued integer coefficient B . It is not difficult to check using Stokes' theorem that if E is trivializable then this action reduces to our first naive action.

The reader may be concerned that this integral depends upon the choice of 4-manifold Z_4 , but a standard argument shows that it does not. Given two such manifolds Z_4 and Z'_4 then we can glue them together along their common boundary X_3 producing a closed oriented 4-manifold $(-Z_4) \cup Z'_4$ (here $-Z_4$ denotes reversed orientation). Now the integral of a Chern class over a closed

⁶In fact a well-defined Chern-Simons theory exists for *arbitrary* compact gauge groups without appealing to 4-manifold extensions. If $H_3(BG) = 0$ (which it does for any torus) then a Chern-Simons theory can be constructed directly using results in [Fre]. Even more generally it is shown there that $H_3(BG)$ is at most a finite group, and even then a classical Chern-Simons theory can be constructed by studying $H^4(BG)$.

⁷It is pointed out in [BM05] that any obstruction to such an extension lives in the oriented bordism group $\Omega_3(BG)$ of the classifying space BG . It is also mentioned in [BM05] that for G abelian $\Omega_3(BG) = 0$, hence we will always be able to extend the bundle in this paper.

oriented manifold is an integer N , i.e.

$$\exp\left(\pi i B \int_{(-Z_4) \cup Z'_4} \tilde{F} \wedge \tilde{F}\right) = \exp(\pi i B N) = 1 \quad (3.9)$$

Furthermore this integer is independent of the extending connection \tilde{A} .

On the other hand the LHS is just

$$\exp\left(\pi i B \left(-\int_{Z_4} \tilde{F} \wedge \tilde{F} + \int_{Z'_4} \tilde{F} \wedge \tilde{F}\right)\right) \quad (3.10)$$

Hence

$$\exp\left(\pi i B \int_{Z_4} \tilde{F} \wedge \tilde{F}\right) = \exp\left(\pi i B \int_{Z'_4} \tilde{F} \wedge \tilde{F}\right) \quad (3.11)$$

If in addition we equip X_3 with a spin structure then there exists a compatible extending *spin* 4-manifold Z_4 [BM05]. In that case the integral of the second Chern class is an *even* integer. Hence the action is well defined even if we allow B to be *any arbitrary integer*. So spin theories are classified by arbitrary integers, and ordinary theories are classified by *even* integers.

Now we generalize to the case where the gauge group is $G = U(1)^N$. Consider a principal $U(1)^N$ -bundle $P \rightarrow X_3$ equipped with a connection Θ . For each projection $\pi_\alpha : U(1)^N \rightarrow U(1)$ where $\alpha \in 1, \dots, N$ we can construct a $U(1)$ bundle P^α via a similar technique that is used to construct an associated bundle. That is, in general if $G \rightarrow H$ is a group equipped with a homomorphism into H then a principal G -bundle P induces a principal H bundle Q . This can be constructed by considering the space $P \times H$ modded out by the right G action $(p, h) \mapsto (p \cdot g, g^{-1} \cdot h)$. Furthermore if P is equipped with a connection Θ then there is an induced connection Θ_Q on Q .

In our case for each α factor $U(1)$ in $U(1)^N$ we have a principal $U(1)$ -

bundle P^α equipped with a connection Θ_α .

A real irreducible representation of $U(1)^N$ is a two-dimensional vector space where each $U(1)$ factor acts as an independent $U(1)$ irreducible representation (all commuting with each other). This representation induces an associated bundle $E \rightarrow X$ and a covariant derivative D .

On the other hand, since each α factor acts as an independent $U(1)$ irreducible representation, we have that each P^α also has a rank two associated bundle $E^\alpha \rightarrow X_3$ with a covariant derivative D^α on E^α . Denote the resulting curvature 2-forms by F^α .

Then it is not difficult to show using similar reasoning to that used above (see page 8 of [BM05]) that such actions are given by extensions to a $U(1)^N$ -bundle $\tilde{E} \rightarrow Z_4$. Such a bundle decomposes as the direct sum of N $U(1)$ -subbundles $\tilde{E}^\alpha \rightarrow Z_4$ where α is an index counting from 1 to N . Let \tilde{F}^α denote the curvature 2-form of \tilde{E}^α . Since each \tilde{E}^α is a $U(1)$ -bundle we have that \tilde{F}^α is valued in $i\mathbb{R}$ (so we leave out any traces that should appear). Consider the action

$$\exp\left(\pi i B_{\alpha\beta} \int_{Z_4} \tilde{F}^\alpha \wedge \tilde{F}^\beta\right) \quad (3.12)$$

where B is a nondegenerate symmetric matrix (B is necessarily symmetric since \tilde{F}^α commutes with \tilde{F}^β). Expanding this becomes

$$\exp\left(\pi i \left(B_{\alpha\alpha} \int_{Z_4} \tilde{F}^\alpha \wedge \tilde{F}^\alpha + \sum_{\alpha < \beta} 2B_{\alpha\beta} \int_{Z_4} \tilde{F}^\alpha \wedge \tilde{F}^\beta \right)\right) \quad (3.13)$$

In light of what we saw for the $U(1)$ case it is necessary that each diagonal entry in B be an *even integer*. Since an extra factor of 2 already appears in the second term from double-counting the off-diagonal entries are arbitrary

integers. In other words B is an integer-valued nondegenerate symmetric bilinear form with even integers along the diagonal. We will call such a symmetric bilinear form *even*.

It is worth noting that if we endow X_3 with a spin structure and extend to a compatible spin 4-manifold Z_4 then the diagonal entries can also be *arbitrary* integers.

Every nondegenerate symmetric bilinear integral form B (not necessarily even) can be thought of as the inner product on a lattice. Summarizing these results is the following proposition [BM05]:

Proposition 3.14. *Classification of classical toral Chern-Simons*

1. *The set of ordinary classical toral Chern-Simons theories is in one-to-one correspondence with even lattices (Λ, B) .*
2. *The set of spin classical toral Chern-Simons theories is in one-to-one correspondence with arbitrary lattices (Λ, B) .*

3.2 Quantization of lattices

In the previous section we have seen that an abelian classical Chern-Simons theory (including a spin theory) is determined by an integer lattice Λ equipped with a symmetric bilinear form $B : \Lambda \times \Lambda \rightarrow \mathbb{Z}$.

Since we are not interested in the general spin case for now we mostly limit our discussion to *even* symmetric bilinear forms. In basis-independent language we mean symmetric bilinear forms B such that $B(X, X) \in 2\mathbb{Z}$ for every $X \in \Lambda$.

It will happen that the canonical quantization program described in section (3.3) will rely heavily on the aspects of lattices described here. We abusively call this “quantization of lattices”. The easiest piece of data that can be harvested from a lattice (Λ, B) (even or not) is the signature $C \in \mathbb{Z}$ of the bilinear form.

For the remaining data we require the following definition:

Definition 3.15. Let R be a ring. A nondegenerate R -valued **quadratic form** on an abelian group (e.g. a lattice) is a function $Q : \Lambda \rightarrow R$ such that:

- $Q(X + Y) - Q(X) - Q(Y) + Q(0)$ defines a bilinear and nondegenerate symmetric form
- We say that Q is a **pure quadratic form** if $Q(nX) = n^2Q(X)$ for every integer n (in particular $Q(0) = 0$).

In this paper if we accidentally drop the “pure” modifier than we still mean pure - we will explicitly say “generalized” otherwise.

Any even lattice (Λ, B) induces a *pure quadratic form* $Q : \Lambda \rightarrow \mathbb{Z}$ given by the formula (division by 2 makes sense because B is even)

$$Q(x) = \frac{1}{2}B(X, X) \tag{3.16}$$

We note that (for even lattices) the pure quadratic form and the bilinear form determine each other: given a pure quadratic form Q a bilinear form can be recovered with the formula

$$B(X, Y) = Q(X + Y) - Q(X) - Q(Y) \tag{3.17}$$

Q is a *pure quadratic refinement* of B .

Discriminant group

From an arbitrary lattice (which determines a classical theory) we construct a finite abelian group \mathcal{D} (the *discriminant group*). The bilinear form B descends to a bilinear form $b : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$, and if the lattice is even then the pure quadratic form Q on Λ descends to a pure quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ as well [Nik80].

The content of the work of Belov and Moore is that quantum toral Chern-Simons theory is (almost) completely determined by (\mathcal{D}, q) , i.e. we have a quantization map

$$\text{Ordinary classical Chern-Simons} \rightarrow \text{Ordinary quantum Chern-Simons} \quad (3.18)$$

that is encoded in the map

$$\text{Even lattice } (\Lambda, B) \rightarrow \text{Discriminant Group } (\mathcal{D}, q, c) \quad (3.19)$$

where $c \equiv C \pmod{24}$ (C is the signature of the bilinear form B). The above map is surjective, however it is not injective.^{8 9}

The construction of the group is as follows: consider the dual lattice Λ^* . Since we have a nondegenerate symmetric bilinear form B we have an embedding of the lattice into its dual $\Lambda \xrightarrow{f} \Lambda^*$ given by $X \mapsto B(X, \cdot)$. In general this map is not invertible over the integers (e.g. it is not possible to

⁸There is a slight error in the main theorem of [BM05]. See appendix (A).

⁹It is important to note that, in contrast to a lattice, a quadratic form on a finite group supplies more information than a bilinear form.

invert the 1×1 matrix $B = (2)$ over the integers) but it can be inverted over the rationals. So let $V = \Lambda \otimes \mathbb{Q}$ and $V^* = \Lambda^* \otimes \mathbb{Q}$ be vector spaces that contain Λ and Λ^* , respectively.

In this case f is invertible and hence we have the (restricted) map $f^{-1} : \Lambda^* \subset V^* \rightarrow V$. It is easy to see that Λ is in the image of Λ^* , so we can think of Λ as a sublattice of Λ^* (all embedded in V). From now on we will think of both Λ and Λ^* as being embedded in V . The finite abelian group is just the quotient $\mathcal{D} = \Lambda^*/\Lambda$.

It is straightforward to check that the bilinear form $B : V \times V \rightarrow \mathbb{Q}$ descends to a (nondegenerate, symmetric) bilinear form $b : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ and that, if the lattice is even, the pure quadratic form $Q : V \rightarrow \mathbb{Q}$ also descends to a pure quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$.

Example 3.20. As an example, consider the rank 1 lattice $\Lambda = \mathbb{Z}$ equipped with the bilinear form $B = (4)$. So $B(1,1) = 4$ and, since this is an even lattice, $Q(1) = \frac{4}{2} = 2$. Tensoring over \mathbb{Q} we see that Λ consists of the numbers

$$\Lambda = \{\dots, -1, 0, 1, 2, 3, \dots\} \tag{3.21}$$

and Λ^* (through the map f^{-1}) consists of the fractions

$$\Lambda^* = \{\dots, -1/4, 0, 1/4, 1/2, 3/4, 1, 5/4, \dots\} \tag{3.22}$$

The discriminant group \mathcal{D} is just

$$\mathcal{D} = \{0, 1/4, 1/2, 3/4\} \cong \mathbb{Z}_4 \tag{3.23}$$

The induced bilinear form is just

$$b(1/4, 1/4) = B(1/4, 1/4) \pmod{1} = 1/4 * 4 * 1/4 \pmod{1} = 1/4 \pmod{1} \quad (3.24)$$

and the induced quadratic form is

$$q(1/4) = \frac{1}{2}B(1/4, 1/4) \pmod{1} = 1/8 \pmod{1} \quad (3.25)$$

The value of b and q on the generator $1/4$ determines all of the values completely.¹⁰

Since the rank of the lattice (here rank $N = 1$) is just the rank of the original gauge group $U(1)^N$ we say that the above example is “ $U(1)$ Chern-Simons at level $B = 4$ ”. Obviously the “level” becomes a matrix in higher rank.

Example 3.28. Let us consider another example. For this let us forget the lattice and just consider the same finite abelian group

$$\mathcal{D} = \{0, 1/4, 1/2, 3/4\} \cong \mathbb{Z}_4 \quad (3.29)$$

We keep the same bilinear form

$$b(1/4, 1/4) = 1/4 \pmod{1} \quad (3.30)$$

¹⁰True since b is bilinear and q is pure. For a choice of generator x any two arbitrary elements can be written as nx and mx for integers n and m . Hence

$$b(nx, mx) = mnb(x, x) \quad (3.26)$$

and

$$q(nx) = n^2q(x) \quad (3.27)$$

but use a *different* pure quadratic refinement

$$q(1/4) = 5/8 \pmod{1} \tag{3.31}$$

(we obtained this quadratic form by taking the value of the previous quadratic form on the generator and adding $1/2$). It is easy to verify that this pure quadratic form is a refinement of b . This is clearly *not* $U(1)$ at level 4. It is also not clear that this data lifts to a lattice.¹¹

So for $\mathcal{D} = \mathbb{Z}_4$ and the same bilinear form we have found two distinct pure quadratic refinements.

Example 3.32. Consider a rank 1 lattice $\Lambda = \mathbb{Z}$ with bilinear form $B = (3)$. This lattice is not even, so it does not induce a pure quadratic refinement. The discriminant group is

$$\mathcal{D} = \{0, 1/3, 2/3\} \cong \mathbb{Z}_3 \tag{3.33}$$

and the induced bilinear form is

$$b(1/3, 1/3) = B(1/3, 1/3) \pmod{1} = 1/3 * 3 * 1/3 = 1/3 \pmod{1} \tag{3.34}$$

As stated, a pure quadratic form is *not* induced by this lattice.

However, if we disregard the classical lattice and simply consider the group $\mathcal{D} = \mathbb{Z}_3$ equipped with the bilinear form b as above then we can produce

¹¹However, we will see below that it does. All pure quadratic forms on finite abelian groups will be realized by *even* lattices.

a pure quadratic refinement of b :

$$q(1/3) = 2/3 \pmod{1} \tag{3.35}$$

$$q(2/3) = 1/3 \pmod{1} \tag{3.36}$$

$$q(0) = 0 \pmod{1} \tag{3.37}$$

Again, it is enough to specify q on the generator, but we list all of the values explicitly for clarity. It is routine to verify that this pure quadratic form is a refinement of b . It is also straightforward to check that this is the *unique* pure quadratic form that is compatible with b (see lemma (3.38)).

However, this theory is *not* $U(1)$ at level 3 (the first part of this example) since that lattice did not induce a pure quadratic form (it is not an even theory). The theory described here, however, can be lifted (as we shall see) to a different (greater rank) even lattice since q is pure.

By studying these two examples and considering the possible bilinear forms and corresponding pure quadratic refinements on an arbitrary cyclic group we have the following proposition (which is clearer if the readers prove it for themselves)

Lemma 3.38. *Let \mathcal{D} be a cyclic group \mathbb{Z}_N equipped with a symmetric bilinear form $b : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ (possibly degenerate). Then:*

1. *If $b = 0$ then $q = 0$ identically.*
2. *If $b \neq 0$ and N is even then there are exactly two pure quadratic refinements of b (on a generator x we have either $q_0(x)$ or $q_1 = q_0(x) + \frac{1}{2}$).*

3. If $b \neq 0$ and N is odd then there is a unique pure quadratic refinement of b .

Proof. Pick a generator x for \mathcal{D} . Since $Nx \equiv 0$ we have that

$$b(x, x) = \frac{m}{N} \tag{3.39}$$

for some integer $m < N$. Since q is pure we have that

$$b(x, x) = q(x+x) - q(x) - q(x) = q(2x) - 2q(x) = 4q(x) - 2q(x) = 2q(x) \tag{3.40}$$

so $q(x) = \frac{1}{2}b(x, x)$. Hence we are left to consider the ambiguity when dividing by 2 in \mathbb{Q}/\mathbb{Z} .

If N is even then we obtain two possibilities for q on a generator x :

$$q(x) = \frac{m}{2N} \text{ or } \frac{m}{2N} + \frac{1}{2} \tag{3.41}$$

It is easy to verify that both of these options are well defined (i.e. $q(Nx) = 0$). The value on an arbitrary element nx is defined by asserting purity $q(nx) = n^2q(x)$.

If N is odd then having a $2N$ in the denominator does not produce a well-defined pure quadratic form. Since N is odd there exists instead a unique integer $m' \bmod N$ such that

$$2m' \equiv m \pmod{N} \tag{3.42}$$

So we define $q(x) = m'/N$.

It is also straightforward to check that a different choice of generator x

gives back one of these examples (hint: write the new generator in terms of the old). \square

In particular, since an arbitrary finite abelian group can be decomposed (not uniquely!) as a direct sum of cyclic groups of prime power order we have

Lemma 3.43. *Any arbitrary finite abelian group \mathcal{D} equipped with a symmetric bilinear form b (perhaps degenerate) admits a pure quadratic refinement.*

Proof. Choose a decomposition of \mathcal{D} into cyclic groups. Each cyclic factor \mathbb{Z}_{N_i} considered by itself has a (possibly degenerate) symmetric bilinear form b_i which is just the restriction of b to \mathbb{Z}_{N_i} . By lemma (3.38) choose a pure quadratic refinement q_i .

Now we must combine the q_i 's into a pure quadratic refinement q defined on the whole group. Given an element of the form $x + y \in \mathcal{D}$ where x is in one factor i and y is in another j define

$$q(x + y) := b(x, y) + q_i(x) + q_j(y) \tag{3.44}$$

It is easy to see that this is the only possibility (and that q is pure). \square

The existence of a pure quadratic refinement will be useful in the sequel.

Gauss sums (reciprocity)

We hinted above in equation (3.19) that we must manually keep around information about the signature C of B when we quantize since passing to the discriminant group “loses memory” of the signature (for our purposes we actually only need to keep the value of $c = C \bmod 24$).

However some information about C is maintained in (\mathcal{D}, q) alone. Gauss proved a relation (a *Gauss sum* or *reciprocity*) on rank 1 even lattices that has since been extended to arbitrary even lattices. For reference see Milnor and Husemoller [MH73] (especially the appendix. We note that the majority of the book applies to unimodular lattices only, i.e. $\det B = \pm 1$). Other references include Nikulin [Nik80]

In fact the induced quadratic refinement q on \mathcal{D} can reproduce information about the signature (but only mod 8) according to the formula

$$\frac{1}{\sqrt{|\mathcal{D}|}} \sum_{x \in \mathcal{D}} \exp(2\pi i q(x)) = \exp(2\pi i C/8) \quad (3.45)$$

Example 3.46. Consider again example (3.20) which is $U(1)$ at level 4. Computing the Gauss sum gives $C \equiv 1 \pmod{8}$ which agrees with expectation since this theory arises from a rank 1 lattice equipped with a bilinear form with signature 1.

Now consider example (3.28) which was a theory *different* from $U(1)$ at level 4. From Gauss sum considerations we see that, if the theory is realized by an even lattice (which it is), then the signature of the lattice mod 8 is $C \equiv 5 \pmod{8}$ (clearly not a rank 1 lattice).

Generalized quadratic forms and spin theories

Let us return momentarily to arbitrary (not necessarily even) lattices (Λ, B) . Although spin theories are not the subject of this paper, we wish to clarify for ourselves some of the constructions that are discussed in [BM05]. In addition we make explicit some observations that are not mentioned there.

We have seen that we have a quantization map encoded in the map

$$\text{Even lattice } (\Lambda, B) \rightarrow \text{Discriminant Group } (\mathcal{D}, q, c) \quad (3.47)$$

However, [BM05] specifies a quantization for *arbitrary* lattices, so we should have a more general map

$$\text{Lattice } (\Lambda, ?_1) \rightarrow \text{Discriminant Group } (\mathcal{D}, ?_2, c) \quad (3.48)$$

It is not immediately clear what should play the role of $?_1$ and $?_2$. Let us describe the construction.

It is easy to see that for any symmetric nondegenerate bilinear form B (even or not) on Λ there exists an element $W \in \Lambda^*$ such that $B(X, X) = B(X, W) \pmod{2}$ for every $X \in \Lambda$. In fact, if W satisfies this then it is trivial to show that $W + 2\lambda$ does as well for any $\lambda \in \Lambda^*$. Conversely, since B is nondegenerate it is also trivial to see that if W and W' satisfy the condition then $W' = W + 2\lambda$ for some $\lambda \in \Lambda^*$.

In other words there exists a *unique class* $[W] \in \Lambda^*/2\Lambda^*$ such that $B(X, X) = B(X, [W]) \pmod{2}$ for every $X \in \Lambda$. Such a class is called the *characteristic class* [BM05] or the *Wu class* [Del99] for the lattice (Λ, B) . We call a specific choice of W in Λ^* a *Wu representative*.

As a special case if the lattice is even then (by definition) $B(X, X) = 0 \pmod{2}$ for every $X \in \Lambda$, hence $[W] = [0] \in \Lambda^*/2\Lambda^*$. Conversely if $[W] = [0]$ then the lattice is even. Since one of the representatives of $[0]$ is just the identity element $W = 0 \in \Lambda^*$ we have - in the case of even lattices - a canonical choice $W = 0$ picked out. For odd lattices there is no such distinguished

representative.

So for even lattices (that we have already considered) the construction that follows momentarily reduces to a single pure quadratic form by setting $W = 0$. For the general theory there will be no preferred representative, hence no preferred *generalized* quadratic form; we will be forced to be content with an equivalence class of (generalized) quadratic forms on \mathcal{D} .

Let us start with a definition:

Definition 3.49. Let q and q' be two \mathbb{Q}/\mathbb{Z} -valued generalized quadratic forms on a finite abelian group \mathcal{D} . Then we say that q is equivalent to q' if there exists a fixed $\delta \in \mathcal{D}$ such that $q'(x) = q(x - \delta)$ for every $x \in \mathcal{D}$.

Now finally we are ready to construct a set of generalized quadratic forms. Consider a lattice (Λ, B) where generically B is odd. Consider the induced discriminant group \mathcal{D} and the induced bilinear form b . Since B is generically odd we do not have an induced pure quadratic form.

From the lattice (which defines a Wu class $[W]$) we need an algorithm to construct a generalized quadratic form $Q : V \rightarrow \mathbb{Q}$ that descends to a well-defined generalized quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$. Let all of the Wu representatives of $[W]$ be denoted by $\{W_i\}_{i \in \mathbb{Z}}$. Since we have infinitely-many representatives W_i we will not be able to construct a single quadratic form, but rather a family of quadratic forms (we shall see momentarily why this constant term is used):

$$Q_{W_i}(X) := \frac{1}{2}B(X, X - W_i) + \frac{1}{8}B(W_i, W_i) \quad (3.50)$$

Each Q_{W_i} descends to a well-defined generalized quadratic form on \mathcal{D}

$$q_i(x) := \frac{1}{2}B(X, X - W_i) + \frac{1}{8}B(W_i, W_i) \pmod{1} \quad (3.51)$$

where $X \in \Lambda^*$ is an arbitrary lift of $x \in \mathcal{D}$ (the choice of lift does not affect the value of the form because of the defining property for W_i).

It is routine to verify that each q_i is a generalized quadratic *refinement* of b (i.e. $q_i(x + y) - q_i(x) - q_i(y) + q_i(0) = b(x, y)$).

Perhaps more interesting, if $W_i, W_j \in \Lambda^*$ are two Wu representatives of $[W]$ then it is easy to show (using that fact that $W_i = W_j + 2\lambda$ for some $\lambda \in \Lambda^*$) that the generalized quadratic refinements q_i and q_j are *equivalent* in the sense defined above.

Even further, it is a simple calculation to show that an entire equivalence class of generalized quadratic forms is realized by the set of all Wu representatives $\{W_i\}_{i \in \mathbb{Z}}$. So $[W]$ determines completely an equivalence class of generalized quadratic refinements which we denote by

$$[q_W] \quad (3.52)$$

Now we know exactly what to substitute for $?_1$ and $?_2$ in the more general quantization map above:

$$Lattice(\Lambda, \{Q_{W_i}\}) \rightarrow Discriminant\ Group(\mathcal{D}, [q_W], c) \quad (3.53)$$

It is easy to see that this map reduces to the old quantization map defined only on even lattices (where $[W] = [0]$) by picking the special pure quadratic refinement defined by $W = 0$ out of the equivalence class.

The reason for choosing the constant term as in equation (3.51) is that then the Gauss reciprocity formula generalizes to arbitrary generalized quadratic forms (see pg 70 in Hopkins and Singer [MH02]). Hence partial information (mod 8) about the signature of B is retained in the same formula

$$\frac{1}{\sqrt{|\mathcal{D}|}} \sum_{x \in \mathcal{D}} \exp(2\pi i q_i(x)) = \exp(2\pi i C/8) \quad (3.54)$$

Obviously different q_i 's in the same equivalence class give the same number on the LHS, hence define the same $C \bmod 8$.

The quantization map is surjective

The “lattice quantization” map in equation (3.53) is surjective. However the map is *not* injective (in fact infinitely-many classical theories will map onto a given quantum theory).

Consider an arbitrary finite abelian group \mathcal{D} equipped with an equivalence class of nondegenerate generalized quadratic forms $[q]$. Use the Gauss sum formula (equation (3.54)) to define a “signature” integer $C \bmod 8$. The term “signature” doesn’t technically make sense because there is no classical lattice here, but we use it anyway. $C \bmod 8$ is determined by \mathcal{D} and $[q]$, so it is not extra information.

However, we require not just an integer mod 8, but rather an integer mod 24. So suppose that, in addition, we are given an integer $c \bmod 24$ such that $c \equiv C \bmod 8$. Obviously for a given C there are only 3 possibilities for such a c .

Then we can ask the following question: does the trio of data $(\mathcal{D}, [q], c)$

lift to a classical lattice? ¹² The answer is yes. We shall start with the simpler case (which is the only one relevant for the remainder of this paper).

We know that an even lattice (Λ, B) maps under equation (3.19) to a trio (\mathcal{D}, q, c) where q is a *pure* nondegenerate quadratic form and c is an integer mod 24 that satisfies the Gauss sum in equation (3.54).

On the other hand, given such a trio $\{ (\mathcal{D}, q, c)$ where \mathcal{D} is a finite abelian group, q is a nondegenerate pure quadratic form, and c is an integer mod 24 that satisfies the Gauss formula $\}$ can this be lifted to an even lattice (Λ, B) ? The following result answers this positively (corollary 1.10.2 pg 117 in [Nik80]):

Corollary 3.55. (*V.V. Nikulin, 1979*) *Let $r_+ \geq 0$ and $r_- \geq 0$ be integers. Consider a finite abelian group \mathcal{D} equipped with a \mathbb{Q}/\mathbb{Z} -valued nondegenerate pure quadratic form q . Define the “signature” mod 8 of q by the Gauss sum formula in equation (3.54). Then if the quantity $r_+ + r_-$ is sufficiently large and if $r_+ - r_- \equiv \text{sign } q \pmod{8}$ then there exists an even lattice (Λ, B) such that*

1. (\mathcal{D}, q) is the discriminant group and quadratic form from (Λ, B)
2. (Λ, B) has r_+ positive eigenvalues and r_- negative eigenvalues

Nikulin’s original statement provides estimates on “sufficiently large”, but we do not need them. Note that the modifier “pure” is left out of Nikulin’s version because in [Nik80] all quadratic forms are *defined* to be pure.

As can be seen, a given trio lifts to infinitely-many even lattices. We conclude that the even quantization map in equation (3.19) is surjective but

¹²We note that $[q]$ determines a bilinear form b , hence we could write the data as a quartet $(\mathcal{D}, b, [q], c)$.

not injective.

Now consider a trio $\{ (\mathcal{D}, [q], c) \}$ where \mathcal{D} is a finite abelian group, $[q]$ is an equivalence class of nondegenerate generalized quadratic forms, and c is an integer mod 24 that satisfies the Gauss formula $\}$. Can this be lifted to a (generically odd) lattice? Consider Nikulin's results about odd lattices (Corollary 1.16.6 [Nik80]):

Corollary 3.56. *(V.V. Nikulin, 1979) Let $r_+ \geq 0$ and $r_- \geq 0$ be arbitrary positive integers. Consider a finite abelian group \mathcal{D} equipped with a \mathbb{Q}/\mathbb{Z} -valued nondegenerate symmetric bilinear form b . Then if the quantity $r_+ + r_-$ is sufficiently large then there exists a (possibly odd) lattice (Λ, B) such that*

1. (\mathcal{D}, b) is the discriminant group and bilinear form from (Λ, B)
2. (Λ, B) has r_+ positive eigenvalues and r_- negative eigenvalues

Again what we present here is weaker than the corollary presented in the original work.

This corollary shows that the data (\mathcal{D}, b, c) lifts to a (possibly odd) lattice (Λ, B) where signature $B = C = r_+ - r_- \equiv c \pmod{24}$ for arbitrary integer $c \pmod{24}$. Note the appearance of b rather than $[q]$ in the trio here. This indicates that the bilinear form lifts, but we have still not seen that $[q]$ lifts ($[q]$ lifts means that it is derived from the Wu class $[W]$ on the lift lattice). We have not seen the following extension of Nikulin's theorem explicitly stated and proven in the literature, hence we prove it here for completeness:

Proposition 3.57. *The trio $(\mathcal{D}, [q], c)$ lifts to a (possibly odd) lattice.*

Proof. To see that $[q]$ lifts as well let us compare it to $[q_W]$ where $[W]$ is the Wu class of the lifted lattice (Λ, B) . We need to show that $[q] = [q_W]$ so pick

a Wu representative W and consider the induced generalized quadratic form

$$q_W(x) \equiv \frac{1}{2}B(X, X - W) + \frac{1}{8}B(W, W) \pmod{1} \quad (3.58)$$

where $X \in \Lambda^*$ is an arbitrary lift of $x \in \mathcal{D}$. Pick one of the quadratic forms q out of the equivalence class $[q]$ as well. We want to compare q and q_W (their induced bilinear forms b are at least the same because q_W is constructed from a lift of b . Also we have already seen that $C \equiv c \pmod{24}$ by construction of the lift so q and q_W satisfy the Gauss sum formula for the same value of $C \pmod{8}$).

It is easier to compare them if we strip off the constants, so define $\tilde{q}(x) = q(x) - q(0)$ and $\tilde{q}_W(x) = q_W(x) - q_W(0) = \frac{1}{2}B(X, X - W)$. Clearly

$$\tilde{q}(x + y) - \tilde{q}(x) - \tilde{q}(y) = \quad (3.59)$$

$$[q(x + y) - q(0)] - [q(x) - q(0)] - [q(y) - q(0)] = \quad (3.60)$$

$$q(x + y) - q(x) - q(y) + q(0) = b(x, y) \quad (3.61)$$

so the bilinear form is not changed when passing from q to \tilde{q} . A similar statement holds for q_W to \tilde{q}_W .

Since \tilde{q} and \tilde{q}_W refine the same bilinear form b they differ by a linear term. This can be seen from

$$[\tilde{q} - \tilde{q}_W](x + y) - [\tilde{q} - \tilde{q}_W](x) - [\tilde{q} - \tilde{q}_W](y) = [b - b](x, y) = 0 \quad (3.62)$$

which shows that $[\tilde{q} - \tilde{q}_W]$ is linear. But b is nondegenerate so any linear function is of the form $b(x, \delta)$ for some fixed $\delta \in \mathcal{D}$. So

$$[\tilde{q} - \tilde{q}_W](x) = b(x, \delta) = B(X, \Delta) \pmod{1} \quad (3.63)$$

for some fixed $\delta \in \mathcal{D}$ ($\Delta \in \Lambda^*$ is an arbitrary lift of $\delta \in \mathcal{D}$). Therefore

$$\tilde{q}(x) = q_{\tilde{W}} + B(X, \Delta) \bmod 1 \quad (3.64)$$

$$= \frac{1}{2}B(X, X - W) + B(X, \Delta) \bmod 1 \quad (3.65)$$

$$= \frac{1}{2}B(X, X - (W - 2\Delta)) \bmod 1 \quad (3.66)$$

The last line is of the form $q_{\tilde{W}'}$ where $W' = W - 2\Delta$ is just another choice of representative for the same Wu class $[W]$.

So we see that $\tilde{q} = q_{\tilde{W}'}$. Now all that we need to do is put the constants back in. We need to check if

$$q(x) = \tilde{q}(x) + q(0) \quad (3.67)$$

equals

$$q_{W'}(x) = q_{\tilde{W}'}(x) + \frac{1}{8}B(W', W') \bmod 1 \quad (3.68)$$

$$= \frac{1}{2}B(X, X - W') + \frac{1}{8}B(W', W') \bmod 1 \quad (3.69)$$

Now it is clear that since $q_{W'}$ is in the same equivalence class as q_W (since W and W' are just different representatives for the same Wu class) they both satisfy the Gauss sum (equation (3.54)) for the same value of $C \bmod 8$.

On the other hand we already mentioned that q and q_W also satisfy the Gauss sum for the same value of $C \bmod 8$ (by the lift construction). Hence they all satisfy the Gauss sum for the same value of $C \bmod 8$. Now the Gauss sum can be viewed as a constraint that determines the constants (because when we stripped off the constants we showed that \tilde{q} equals $q_{\tilde{W}'}$). In this case

we have no choice but to conclude $q(0) = \frac{1}{8}B(W', W') \pmod{1}$.

Summarizing, $q = q_{W'}$ for some Wu representative W' , hence the equivalence class of quadratic refinements $[q]$ actually lifts through the Nikulin construction (to $[q_W]$). We conclude that the trio $(\mathcal{D}, [q], c)$ lifts. \square

3.3 Canonical quantization of Belov and Moore

In the last section we discussed the quantization of lattices. We use the term quantization since the resulting trio of data (\mathcal{D}, q, c) encodes the quantization of toral (spin or non-spin) Chern-Simons gauge theory. In this section we transcribe the relevant Hilbert space structure that arises from the wavefunctions constructed in [BM05] and recall that this provides a (non-extended) 2-d modular functor (see chapter (2)).

Hilbert space preliminaries

First it is useful to mention some preliminaries before reproducing the action of the mapping class group for *closed* surfaces¹³ on the Hilbert space of wavefunctions as described in section 5.6 of [BM05].

Following Belov and Moore we avoid the special considerations that must be taken into account when the surface Σ is the Riemann sphere (see chapter (2)) and skip to the case where Σ is a *closed* oriented Riemann surface with genus $g \geq 1$.

Let us pick a *canonical basis* for the first homology group $H_1(\Sigma, \mathbb{Z})$, i.e.

¹³Note that Belov and Moore study only *fixed* vortices (marked arcs, or colored boundary circles). The braiding and twisting of such quasiparticles must also be described to specify an extended 2-d modular functor (see chapter (2)). Hence we restrict our attention to closed surfaces.

an *ordered* set of loops $\{a_i, b_i\}_{i=1, \dots, g}$ in Σ such that the oriented intersection numbers are given by

$$I(a_i, b_j) = -I(b_j, a_i) = \delta_{ij} \tag{3.70}$$

$$I(a_i, a_j) = 0 \tag{3.71}$$

$$I(b_i, b_j) = 0 \tag{3.72}$$

Such a basis always exists (but is not unique) for any closed Riemann surface Σ .¹⁴ Clearly this intersection matrix defines a symplectic inner product on $H_1(\Sigma, \mathbb{Z})$.

Orientation-preserving diffeomorphisms map loops to loops *and preserve intersection numbers*, hence on the canonical basis $\{a_i, b_i\}$ the mapping class group $\text{MCG}(\Sigma)$ acts via invertible integer-valued matrices *that leave the symplectic inner product matrix unchanged*. Such matrices are elements of the (integral) symplectic group $\text{Sp}(2g, \mathbb{Z})$. So we have a map

$$\text{MCG}(\Sigma) \rightarrow \text{Sp}(2g, \mathbb{Z}) \tag{3.73}$$

In general this map is surjective and the kernel is the Torelli group. It is claimed in [BM05] that for the abelian theories considered there the Torelli group acts trivially. In other words the mapping class group action on the wavefunctions is encoded entirely in $\text{Sp}(2g, \mathbb{Z})$ for abelian theories.

Since $H_1(\Sigma, \mathbb{Z})$ is $2g$ -dimensional let us write the choice of canonical

¹⁴This *choice* of canonical basis is a variant of the *extra structure* that is required on Σ in order to define an anomaly-free TQFT. See chapter (2). Also we shall not bother to distinguish between homology *classes* and representative loops.

basis using the convention

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots a_g = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad b_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots b_g = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad (3.74)$$

The symplectic group is then generated by matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1,t} \end{pmatrix}, \quad A \in \mathrm{GL}(g, \mathbb{Z}), \quad \text{i.e. } \det(A) = \pm 1 \quad (3.75)$$

$$\begin{pmatrix} \mathbb{1}_g & B \\ 0 & \mathbb{1}_g \end{pmatrix}, \quad B \text{ is any symmetric integral } g \times g \text{ matrix}$$

$$\begin{pmatrix} 0 & -\mathbb{1}_g \\ \mathbb{1}_g & 0 \end{pmatrix}$$

As usual in genus $g = 1$ these matrices are $\mathbb{1}$, t , and s - the familiar generators of the modular group $\mathrm{SL}(2, \mathbb{Z}) \cong \mathrm{Sp}(2, \mathbb{Z})$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.76)$$

The chosen canonical basis $\{a_i, b_i\}_{i \in 1, \dots, g}$ for $H_1(\Sigma, \mathbb{Z})$ induces a dual basis $\{\alpha_i, \beta_i\}_{i \in 1, \dots, g}$ of integral 1-forms $H^1(\Sigma, \mathbb{Z})$. This is useful since (chapter (2)) the Kähler quantization procedure has as classical configuration space

the moduli space of flat connections \mathcal{M} (which are essentially 1-forms). The Hilbert space is comprised of wavefunctions of the form Ψ (1-forms).

Using the dual basis $\{\alpha_i, \beta_i\}_{i \in 1, \dots, g}$ we can decompose any 1-form ω ¹⁵ into ¹⁶

$$\omega = \omega_1^i \alpha_i + \omega_2^i \beta_i \quad (3.77)$$

for $\omega_1^i, \omega_2^i \in \mathbb{R}$. The transformations in equation (3.75) are transposed when acting on the dual basis $\{\alpha_i, \beta_i\}_{i \in 1, \dots, g}$

$$\begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad A \in \text{GL}(g, \mathbb{Z}), \quad \text{i.e. } \det(A) = \pm 1 \quad (3.78)$$

$$\begin{pmatrix} \mathbb{1}_g & 0 \\ B^t & \mathbb{1}_g \end{pmatrix}, \quad B \text{ is any symmetric integral } g \times g \text{ matrix} \quad (3.79)$$

$$\begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix} \quad (3.80)$$

(obviously $B^t = B$). The induced action on any wavefunction is given by

1. A transform:

$$(M_A \cdot \Psi)(\omega) := \Psi(M_A \cdot \omega) = \Psi(A^t \cdot \omega_1, A^{-1} \cdot \omega_2) \quad (3.81)$$

2. B transform:

$$(M_B \cdot \Psi)(\omega) := \Psi(M_B \cdot \omega) = \Psi(\omega_1, \omega_2 + B \cdot \omega_1) \quad (3.82)$$

¹⁵The universal coefficient theorem tells us that $H^1(\Sigma, \mathbb{R}) \cong H^1(\Sigma, \mathbb{Z}) \otimes \mathbb{R}$.

¹⁶Warning: our notation diverges from that in [BM05]. We use ω_1 and ω_2 instead of a^1 and a_2 to avoid notation collisions. Our indices are also placed differently.

3. S transform:

$$(M_S \cdot \Psi)(\omega) := \Psi(M_S \cdot \omega) = \Psi(\omega_2, -\omega_1) \quad (3.83)$$

Now let us discuss a few further constructions utilized in [BM05] to write down a *basis* of wavefunctions (and to understand the above group action in terms of this basis).

Dependence on spin structure and Wu class

The basis of wavefunctions depends on the choice of spin structure and choice of Wu class (see below). First, it is a fact that any compact oriented 3-manifold X admits at least one spin structure [Sti00]. This is equivalent to saying that the first and second Stiefel-Whitney classes (which are valued in $H^1(X, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$) for the tangent bundle vanish, i.e. $w_1(TX) = w_2(TX) = 0 \in H^1(X, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$ (in fact TX is trivializable).

The group $H^1(X, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$ itself need not be zero, however. In fact $H^1(\Sigma, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$ enumerates the different possible spin structures on X .¹⁷ Explicitly for a manifold of the form $X = \Sigma \times I$ (as in the current Hamiltonian formulation) we have that $H^1(X, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \cong H^1(\Sigma, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$ since X deformation retracts onto Σ . But by the universal coefficient theorem we see that

$$H^1(\Sigma, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \cong H^1(\Sigma, \mathbb{Z}) \otimes \frac{1}{2}\mathbb{Z}/\mathbb{Z} \quad (3.84)$$

¹⁷The space of spin structures is an $H^1(\Sigma, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$ -torsor. However given our choice of canonical homology basis $\{a_i, b_i\}_{i=1, \dots, g}$ a preferred spin structure is determined (see pg. 27 of [BM05]). We identify this with $0 \in H^1(X, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$ (i.e. we have fixed a preferred origin for the spin structures, and hence the space of spin structures can be identified with $H^1(\Sigma, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$ itself).

Manifestly this has 2^{2g} elements that can be written in terms of the dual basis $\{\alpha_i, \beta_i\}_{i \in 1, \dots, g}$ (but with $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ coefficients).

In light of this let us encode a fixed spin structure by specifying a set of coefficients $([\epsilon_1], [\epsilon_2]) \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$ (i.e. a spin structure is given by $[\epsilon_1] \cdot \alpha + [\epsilon_2] \cdot \beta \in H^1(\Sigma, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$). For this fixed spin structure the main idea is to define a Hilbert space $H_{([\epsilon_1], [\epsilon_2])}$ of wavefunctions using theta functions.

The spin structure $([\epsilon_1], [\epsilon_2])$ is not the only piece of data needed to write down a Hilbert space. Recall from section (3.2) that the “quantization” of a classical lattice is encoded in the data

$$(\mathcal{D}, [q_W], c) \tag{3.85}$$

where \mathcal{D} is a finite abelian group, $[q_W] : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ is an equivalence class of quadratic forms on \mathcal{D} constructed from the Wu class $[W] \in \Lambda^*/2\Lambda^*$ of the classical lattice, and c is an integer mod 24 that is essentially a choice of cube root of the Gauss reciprocity formula. The content of the Belov-Moore construction is that the Hilbert space (and action of the mapping class group) is determined by this data alone. So we add additional decoration to the above Hilbert space

$$H_{([\epsilon_1], [\epsilon_2]), (\mathcal{D}, [q_W], c)} \tag{3.86}$$

or, more compactly

$$H_{([\epsilon_1], [\epsilon_2]), [W]} \tag{3.87}$$

$H_{([\epsilon_1], [\epsilon_2]), [W]}$ can only be explicitly written down by picking a representative $(\epsilon_1, \epsilon_2) \in (\frac{1}{2}\mathbb{Z})^{2g}$ of $([\epsilon_1], [\epsilon_2])$. Likewise, we are forced to pick an explicit representative $W \in \Lambda^*$ from the Wu class $[W]$. Unfortunately the basis of

wavefunctions *does naively depend* on these representative choices, however different bases constructed from different representatives are gauge equivalent by an explicit set of gauge transformations (which we list below). Hence there is no loss in generality when picking representatives (ϵ_1, ϵ_2) and W :

$$H_{(\epsilon_1, \epsilon_2), W} \tag{3.88}$$

As discussed in [BM05] there are precisely $|\mathcal{D}^g|$ basis wavefunctions in $H_{(\epsilon_1, \epsilon_2), W}$ enumerated by $\gamma \in \mathcal{D}^g$ (i.e. there is a copy of the discriminant group \mathcal{D} for each canonical basis loop b_i where $i \in 1, \dots, g$):

$$\Psi_{\gamma, (\epsilon_1, \epsilon_2), W}(\text{1-forms}) \tag{3.89}$$

The transformation laws that map one basis of wavefunctions determined by a choice of representative $(\epsilon_1, \epsilon_2), W$ to another choice are derived at the end of section 5.3 in [BM05] (and more succinctly in equation 5.42 in [BM05]). Recall that we are not considering vortices here.¹⁸ The dependence on W is shown in [BM05], but we shall not need it since the Wu representative is unaltered by the action of the symplectic group. The dependence on representative (ϵ_1, ϵ_2) , however, is necessary in what follows. We have

$$\begin{aligned} \Psi_{\gamma, (\epsilon_1 + n_1, \epsilon_2 + n_2), W} = \\ e^{8\pi i q_W(0)[n_1 \cdot n_2 + \epsilon_1 \cdot n_2 - \epsilon_2 \cdot n_1] + 2\pi i n_2^i [q_W(-\gamma_i) - q_W(\gamma_i)]} \Psi_{\gamma + n_1 \otimes \overline{W}, (\epsilon_1, \epsilon_2), W} \end{aligned} \tag{3.90}$$

where \overline{W} is the projection of W into the discriminant group \mathcal{D} . The repeated index $i = 1, \dots, g$ is summed over, as usual (manifestly $(n_1, n_2) \in \mathbb{Z}^{2g}$).

¹⁸In the language of [BM05] set $c^1 = c_2 = 0$.

The results mentioned in the next subsection show that the action of the mapping class group on the theta functions (as formally described in equations (3.81), (3.82), (3.83)) does *not* preserve the spin structure. In light of this Belov and Moore proposed that the full Hilbert space for the theory must be written as a direct sum over the separate spin structures:

$$H_{[W]} = \bigoplus_{[\epsilon_1], [\epsilon_2] \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{\otimes g}} H_{[\epsilon_1], [\epsilon_2], [W]} \quad (3.91)$$

Action of the mapping class group on theta functions

Using the properties of theta functions (see [BM05]) it is possible to cast the action of the mapping class group (discussed in equations (3.81), (3.82), (3.83)) into new expressions (we add the extra decorations to the wavefunctions from here):

1. A transform:

$$\begin{aligned} (M_A \cdot \Psi_{\gamma, (\epsilon_1, \epsilon_2), W})(\omega) &:= \Psi_{\gamma, (\epsilon_1, \epsilon_2), W}(M_A \cdot \omega) = \\ \Psi_{\gamma, (\epsilon_1, \epsilon_2), W}(A^t \cdot \omega_1, A^{-1} \cdot \omega_2) &= \Psi_{A^t \gamma, (A^t \epsilon_1, A^{-1} \epsilon_2), W}(\omega_1, \omega_2) \end{aligned} \quad (3.92)$$

2. B transform:

$$\begin{aligned} (M_B \cdot \Psi_{\gamma, (\epsilon_1, \epsilon_2), W})(\omega) &:= \Psi_{\gamma, (\epsilon_1, \epsilon_2), W}(M_B \cdot \omega) = \\ \Psi_{\gamma, (\epsilon_1, \epsilon_2), W}(\omega_1, \omega_2 + B \cdot \omega_1) &= e^{2\pi i \phi(B)c/24} e^{4\pi i \epsilon_1^i B^{ii} q_W(0) - 2\pi i B^{ii} [q_W(\gamma_i) - q_W(0)]} \\ &\times e^{-2\pi i \sum_{i < j} B^{ij} b(\gamma_i, \gamma_j)} \Psi_{\gamma, (\epsilon_1, \epsilon_2 - B\epsilon_1 - \frac{1}{2} \text{diag}(B)), W}(\omega_1, \omega_2) \end{aligned} \quad (3.93)$$

3. S transform:

$$\begin{aligned} (M_S \cdot \Psi_{\gamma,(\epsilon_1,\epsilon_2),W})(\omega) &:= \Psi_{\gamma,(\epsilon_1,\epsilon_2),W}(M_S \cdot \omega) = \\ \Psi_{\gamma,(\epsilon_1,\epsilon_2),W}(\omega_2, -\omega_1) &= |\mathcal{D}|^{-g/2} \sum_{\gamma' \in \mathcal{D}^g} e^{2\pi i b(\gamma_i, \gamma'_i)} \Psi_{\gamma',(-\epsilon_2,\epsilon_1),W}(\omega_1, \omega_2) \end{aligned} \quad (3.94)$$

Here $b(\cdot, \cdot) : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ is the bilinear form determined by q_W and $i, j \in 1, \dots, g$ are summed over when the indices are repeated (except the i in $2\pi i$ means $2\pi\sqrt{-1}$ of course). The quantity $\phi(B)$ is an integer determined from the matrix B (see [BM05]).

We will always choose the representative $(\epsilon_1, \epsilon_2) \in (\frac{1}{2}\mathbb{Z})^{2g}$ such that every element $\epsilon_1^i, \epsilon_2^i$ is either 0 or $\frac{1}{2}$ (if the above action on the basis wavefunctions destroys this choice then we can use equation (3.90) to put each element back into this form).

Even (non-spin) theories

For the case of an even (non-spin) topological quantum field theory (see section (3.2)) we can always make the special choice for Wu representative $W = 0$ (the quadratic form q_W is then pure). In this case the spin structure $([\epsilon_1], [\epsilon_2])$ is irrelevant. The basis wavefunctions are written in terms of the theta functions *up to non-trivial normalization factors* (see page 28 in [BM05] and the other references cited there for greater detail):

$$\Psi_{\gamma,(\epsilon_1,\epsilon_2),W}(\omega_1, \omega_2) \sim \Theta_{\Lambda+\gamma}^{\epsilon_1 \otimes W, \epsilon_2 \otimes W}(\omega_1, \omega_2) \quad (3.95)$$

Clearly if we set $W = 0$ then different spin structures $([\epsilon_1], [\epsilon_2])$ produce the same wavefunctions. The full Hilbert space is *not a direct sum* over spin structures as in equation (3.91). Instead there are only $|\mathcal{D}^g|$ basis wavefunctions, and the action of the symplectic group reduces to

1. A transform (even theory):

$$(M_A \cdot \Psi_\gamma)(\omega) = \Psi_{A^t \gamma}(\omega) \quad (3.96)$$

2. B transform (even theory):

$$(M_B \cdot \Psi_\gamma)(\omega) = e^{2\pi i \phi(B)c/24} e^{-2\pi i B^{ii} q_W(\gamma_i)} e^{-2\pi i \sum_{i < j} B^{ij} b(\gamma_i, \gamma_j)} \Psi_\gamma(\omega) \quad (3.97)$$

3. S transform (even theory):

$$(M_S \cdot \Psi_\gamma)(\omega) = |\mathcal{D}|^{-g/2} \sum_{\gamma' \in \mathcal{D}^g} e^{2\pi i b(\gamma_i, \gamma'_i)} \Psi_{\gamma'}(\omega) \quad (3.98)$$

An example in genus 1

In genus 1 the above symplectic group action on the Hilbert space of wavefunctions can be made more explicit. We take this opportunity to correct some slight calculational errors in subsection 5.6.1 of [BM05] for the benefit of the reader.

Denote the matrix elements of an operator \mathcal{O} acting from $H_{\epsilon_1, \epsilon_2, W}$ to $H_{\epsilon'_1, \epsilon'_2, W}$ by the notation $\mathcal{O}_{\gamma'}^\gamma \left[\begin{smallmatrix} 2\epsilon_1 & 2\epsilon_2 \\ 2\epsilon'_1 & 2\epsilon'_2 \end{smallmatrix} \right]$.¹⁹ Then in genus 1 the t and s symplectic matrices induce operators T and S given by the following matrix elements

¹⁹Beware: our primed and unprimed indices are exactly opposite to that in [BM05]. We seek to remain consistent with our previous notation.

(everything not listed is zero):

$$T_{\gamma'}^{\gamma} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = e^{2\pi ic/24 - 2\pi i[q_W(-\gamma) - q_W(0)]} \delta_{\gamma'}^{\gamma} \quad (3.99)$$

$$T_{\gamma'}^{\gamma} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e^{2\pi ic/24 - 2\pi i[q_W(\gamma) - q_W(0)]} \delta_{\gamma'}^{\gamma} \quad (3.100)$$

$$T_{\gamma'}^{\gamma} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = T_{\gamma'}^{\gamma} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = e^{2\pi ic/24 - 2\pi i q_W(-\gamma)} \delta_{\gamma'}^{\gamma} \quad (3.101)$$

The S matrices are

$$S_{\gamma'}^{\gamma} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = S_{\gamma'}^{\gamma} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |\mathcal{D}|^{-1/2} e^{2\pi ib(\gamma, \gamma')} \quad (3.102)$$

$$S_{\gamma'}^{\gamma} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |\mathcal{D}|^{-1/2} e^{2\pi ib(\gamma, \gamma' + \bar{W})} \quad (3.103)$$

$$S_{\gamma'}^{\gamma} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = |\mathcal{D}|^{-1/2} e^{2\pi ib(\gamma, \gamma' + \bar{W}) + 4\pi i q_W(0)} \quad (3.104)$$

For *even* theories the spin labelling collapses since set $W = 0$. Since q_W is then pure we have $q_W(0) = 0$ and $q_W(-\gamma) = q_W(\gamma)$. The resulting T and S operators are much simpler

$$T_{\gamma'}^{\gamma} = e^{2\pi ic/24 - 2\pi i q_W(\gamma)} \delta_{\gamma'}^{\gamma} \quad (3.105)$$

$$S_{\gamma'}^{\gamma} = |\mathcal{D}|^{-1/2} e^{2\pi ib(\gamma, \gamma')} \quad (3.106)$$

Chapter 4

Modular Tensor Categories

4.1 Introduction

The goal of this chapter is to provide a brief sketch of modular tensor categories to lay a foundation for future chapters. Modular tensor categories (MTCs) grew somewhat simultaneously out of the study of conformal field theory by Moore and Seiberg [MS89] and quantum groups by Lusztig, Jimbo, Reshetikhin and Turaev, and others (see the references in [RT90], [RT91], and [KM91] for a more complete listing).

For the majority of this chapter we follow [Tur94] and [BK00] (borrowing conventions and notation from both). Our arrows will be in exactly the opposite direction to those in [Tur94]. We also follow the definition of the S -matrix in [BK00]. We have also found the unpublished notes of Boyarchenko [Boy] useful.

Both books [Tur94] and [BK00] consider in detail *strict* ribbon categories. This is not sufficient for our purposes and hence we shall consider

ribbon categories that are not necessarily strict. However, since strict categories are easier to understand we consider them first in all of the definitions below.

4.2 Monoidal categories

Strict monoidal categories

Definition 4.1. A **strict monoidal category** is a category \mathcal{V} equipped with a covariant bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ and a distinguished object $\mathbb{1}$ such that the following two identities hold:

1. Strict identity:

$$U \otimes \mathbb{1} = \mathbb{1} \otimes U = U \tag{4.4}$$

2. Strict associativity:

$$(U \otimes V) \otimes W = U \otimes (V \otimes W) \tag{4.5}$$

Example 4.6. A simple example of a strict monoidal category is the category $\text{Vect}_{\mathbb{C}}$ of complex vector spaces under the usual tensor product. Here the unit object is $\mathbb{1} = \mathbb{C}$.

¹By *covariant bifunctor* we mean that for any two objects $V, W \in \text{Ob}(\mathcal{V})$ there is an object $V \otimes W \in \text{Ob}(\mathcal{V})$, and for any two morphisms $f : V \rightarrow V'$ and $g : W \rightarrow W'$ there is a morphism $f \otimes g : V \otimes W \rightarrow V' \otimes W'$. Functoriality means that given morphisms $f' : V' \rightarrow V''$, $g' : W' \rightarrow W''$ the following identities are required to be satisfied:

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g) \tag{4.2}$$

$$\text{id}_V \otimes \text{id}_W = \text{id}_{V \otimes W} \tag{4.3}$$

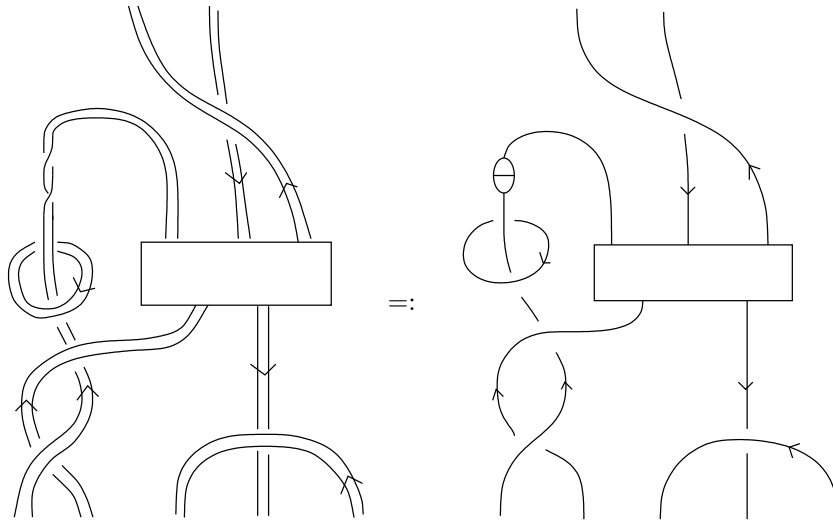


Figure 4.1: A $(k = 5, l = 2)$ -ribbon graph. The diagrammatic presentation is depicted on the right.

Example 4.7. Now we construct a more complicated strict monoidal category Rib_I , called the category of colored ribbon graphs. Here I is some auxiliary set of labels (“colors”).

First we require some preliminary definitions. We will be rather informal here since the following definition is written carefully in [Tur94]:

Definition 4.8. A (k, l) -**ribbon graph** Ω is an *oriented surface* in \mathbb{R}^3 up to *isotopy*. The surface is constructed out of elementary pieces (see figure (4.1)):

1. oriented *ribbons*
2. *coupons*
3. oriented *annuli*

The coupons have two distinguished bases (“in” and “out”) on which the ribbons can terminate. Any ribbon end that terminates on a coupon is *not*

allowed to slide from the “in” side to the “out” side (or vica versa) under isotopy.

Furthermore, we require that for a (k, l) -ribbon graph there are $k \geq 0$ free ribbon ends that are marked as “inputs”, and likewise there are $l \geq 0$ free ribbon ends that are marked as “outputs”. In fact it is always possible to perform an isotopy to put the ribbon graph Ω into a *standard drawing position* (see figure (4.1)), i.e.:

1. The k “input” free ribbon ends are at the bottom. They are ordered from left to right (the ordering can be changed by braiding the free ribbon ends over/under each another).
2. The l “output” free ribbon ends are at the top. They are ordered from left to right.
3. The graph is “face up” (determined by the orientation of Ω) except in finitely-many localized places where the ribbons are *twisted* (see figure (4.2)).
4. The graph sits entirely in the plane of the drawing except at a finite number of overcrossings, undercrossings, and twists (see figure (4.3)).

Because of the standard drawing position it is clear that we can represent any ribbon graph by a **ribbon diagram**, i.e. a diagram where the oriented ribbons are replaced by their oriented cores. The ribbons can be recovered by using the blackboard framing. See the right side of figure (4.1).

Now let I be a set of labels (colors). We define a **colored (k, l) -ribbon graph** as a (k, l) -ribbon graph where each ribbon and each annulus is labeled by some element in I (we do not color the coupons yet).

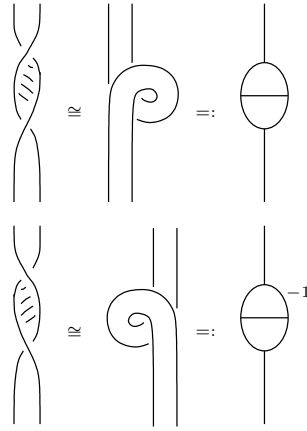


Figure 4.2: On the top is depicted a right twist (a $(1, 1)$ -ribbon graph). On the bottom is depicted a left twist (a $(1, 1)$ -ribbon graph). The diagrammatic presentation is depicted on the right for each.

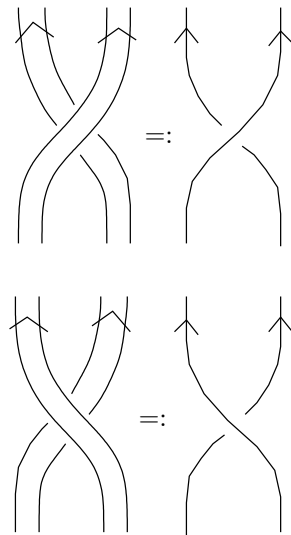


Figure 4.3: On the top is depicted a right braid (a $(2, 2)$ -ribbon graph). On the bottom is depicted a left braid (a $(2, 2)$ -ribbon graph). The diagrammatic presentation is depicted on the right for each.

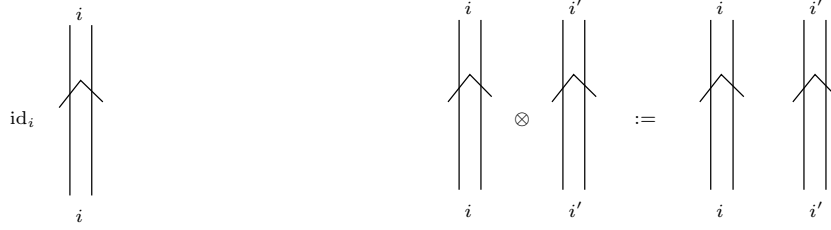


Figure 4.4: The identity morphism $\text{id}_i : [[i, +1]] \rightarrow [[i, +1]]$ in Rib_I is depicted on the left. The tensor product of morphisms in Rib_I (in this case two identity morphisms) is depicted on the right.

Definition 4.9. Define a strict monoidal category Rib_I as follows:

1. The objects are ordered lists $[[i_1, \pm 1], [i_2, \pm 1], \dots]$ where $i_1, i_2, \dots \in I$. The unit object $\mathbf{1}$ is the empty list $[\]$.
2. Given objects $[[i_1, \pm 1], [i_2, \pm 1], \dots, [i_k, \pm 1]]$ and $[[i'_1, \pm 1], [i'_2, \pm 1], \dots, [i'_l, \pm 1]]$ a morphism between them is a colored (k, l) -ribbon graph such that the k “input” ribbons are labeled (in order) by i_1, \dots, i_k and each ribbon is directed up for $+1$ and directed down for -1 . Similarly the l “output” ribbons are labeled by i'_1, \dots, i'_l where they are directed up for $+1$ and down for -1 . It is obvious that these morphisms can be composed by stacking colored ribbon graphs on top of each other.

Rib_I is a strict monoidal category since any two ordered lists can be concatenated

$$[[i_1, \pm 1], [i_2, \pm 1], \dots, [i_k, \pm 1]] \otimes [[i'_1, \pm 1], [i'_2, \pm 1], \dots, [i'_l, \pm 1]] =$$

$$[[i_1, \pm 1], [i_2, \pm 1], \dots, [i_k, \pm 1], [i'_1, \pm 1], [i'_2, \pm 1], \dots, [i'_l, \pm 1]] \quad (4.10)$$

(this defines \otimes on the objects) and ribbon graphs can be placed adjacent

to each other (this defines \otimes on the morphisms - see e.g. the right side of figure (4.4)).

(Non-strict) monoidal categories

We now consider monoidal categories that may not be strict.

Definition 4.11. A **monoidal category** is a category \mathcal{V} equipped with a covariant bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ and a distinguished object $\mathbb{1}$. Furthermore we require a family of natural isomorphisms

$$\{a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)\} \quad (4.12)$$

$$\{r_U : U \otimes \mathbb{1} \rightarrow U\} \quad (4.13)$$

$$\{l_U : \mathbb{1} \otimes U \rightarrow U\} \quad (4.14)$$

such that the following diagrams commute:

Pentagon diagram:

$$\begin{array}{ccc}
 & (U \otimes V) \otimes (W \otimes X) & \\
 & \nearrow^{a_{U \otimes V, W, X}} & \searrow^{a_{U, V, W \otimes X}} \\
 ((U \otimes V) \otimes W) \otimes X & & U \otimes (V \otimes (W \otimes X)) \\
 \downarrow^{a_{U, V, W} \otimes \text{id}_X} & & \uparrow^{\text{id}_U \otimes a_{V, W, X}} \\
 (U \otimes (V \otimes W)) \otimes X & \xrightarrow{a_{U, V \otimes W, X}} & U \otimes ((V \otimes W) \otimes X)
 \end{array} \quad (4.15)$$

Triangle diagram:

$$\begin{array}{ccc}
 (U \otimes \mathbf{1}) \otimes V & \xrightarrow{a_{U,\mathbf{1},V}} & U \otimes (\mathbf{1} \otimes V) \\
 \searrow r_U \otimes \text{id}_V & & \swarrow \text{id}_U \otimes l_U \\
 & U \otimes V &
 \end{array} \tag{4.16}$$

The MacLane Coherence Theorem [Mac97] states that if these commutative diagrams are satisfied then *any* diagram involving a, r, l is commutative, i.e.:

1. given any ordered list A of objects that are tensored together and grouped with parenthesis,
2. and given the same ordered list A' but with different parenthesis grouping (and possibly with unit objects $\mathbf{1}$ appearing/not appearing in different places),
3. then any two ways of getting from A to A' using any combination of the maps a, r, l are the *same*.

This implies in particular that any monoidal category is *monoidal equivalent* (see chapter (5)) to a strict monoidal category.

Example 4.17. There is a straightforward “non-associative” generalization of colored (k, l) -ribbon graphs constructed by Bar-Natan in [BN93], and it is not difficult to construct the corresponding (non-strict) monoidal category Rib_I^{NS} . For example the objects are ordered lists *with parenthesis* $[[i_1, \pm 1], [i_2, \pm 1], \dots]$, and the morphisms are *non-associative* colored (k, l) -ribbon graphs.

4.3 Braided monoidal categories

In this section we define *braided* monoidal categories. The natural setting for the examples in this paper are braided (non-strict) monoidal categories. However, we discuss braided strict monoidal categories first since they are easier to understand.

Braided strict monoidal categories

Definition 4.18. A **braided strict monoidal category** is a strict monoidal category equipped with a family of natural *braiding* isomorphisms

$$\{c_{U,V} : U \otimes V \rightarrow V \otimes U\} \quad (4.19)$$

The braiding isomorphisms represent a *weak* form of commutativity. Note that it is *not* usually true that $c_{V,U} \circ c_{U,V} = \text{id}_{U \otimes V}$. If this condition is satisfied then the category is called **symmetric** (we are interested in non-symmetric categories).

The braiding isomorphisms are required to satisfy the following *hexagon relations*:

$$\begin{array}{ccccc}
 & A \otimes (B \otimes C) & \xrightarrow{c_{A,B \otimes C}} & (B \otimes C) \otimes A & \\
 & \text{dotted} & & \text{dotted} & \\
 & \text{id} & & \text{id} & \\
 (A \otimes B) \otimes C & & & & B \otimes (C \otimes A) \\
 & \text{solid} & & \text{solid} & \\
 & c_{A,B} \otimes \text{id}_C & & \text{id}_B \otimes c_{A,C} & \\
 & & & & \\
 (B \otimes A) \otimes C & \xrightarrow{\text{dotted}} & B \otimes (A \otimes C) & & \\
 & & & &
 \end{array} \quad (4.20)$$

$$\begin{array}{ccccc}
& & (U \otimes V) \otimes W & \xrightarrow{c_{U \otimes V, W}} & W \otimes (U \otimes V) \\
& \swarrow \text{id} & & & \nwarrow \text{id} \\
U \otimes (V \otimes W) & & & & (W \otimes U) \otimes V \\
& \searrow \text{id}_U \otimes c_{V, W} & & & \nearrow c_{U, W} \otimes \text{id}_V \\
& & U \otimes (W \otimes V) & \xrightarrow{\text{id}} & (U \otimes W) \otimes V
\end{array} \tag{4.21}$$

It is easy to check that Rib_I is a braided strict monoidal category (use the braiding graphs as in figure (4.3)). The hexagon relations have a very simple geometric interpretation in Rib_I - it is instructive for the reader to draw them out for himself/herself.

(Non-strict) braided monoidal categories

We now consider braided monoidal categories that may not be strict.

Definition 4.22. A **braided monoidal category** is a monoidal category equipped with a family of natural *braiding* isomorphisms

$$\{c_{U, V} : U \otimes V \rightarrow V \otimes U\} \tag{4.23}$$

In contrast to the strict case the braiding isomorphisms are required to

satisfy more elaborate *hexagon relations*:

$$\begin{array}{ccccc}
& & A \otimes (B \otimes C) & \xrightarrow{c_{A,B \otimes C}} & (B \otimes C) \otimes A \\
& \swarrow & & & \nwarrow \\
(A \otimes B) \otimes C & & & & B \otimes (C \otimes A) \\
& \searrow & & & \swarrow \\
& & (B \otimes A) \otimes C & \xrightarrow{a_{B,A,C}} & B \otimes (A \otimes C)
\end{array}
\quad (4.24)$$

$a_{A,B,C}^{-1}$ (left arrow), $a_{B,C,A}^{-1}$ (right arrow), $c_{A,B} \otimes \text{id}_C$ (bottom-left arrow), $\text{id}_B \otimes c_{A,C}$ (bottom-right arrow)

$$\begin{array}{ccccc}
& & (U \otimes V) \otimes W & \xrightarrow{c_{U \otimes V,W}} & W \otimes (U \otimes V) \\
& \swarrow & & & \nwarrow \\
U \otimes (V \otimes W) & & & & (W \otimes U) \otimes V \\
& \searrow & & & \swarrow \\
& & U \otimes (W \otimes V) & \xrightarrow{a_{U,W,V}^{-1}} & (U \otimes W) \otimes V
\end{array}
\quad (4.25)$$

$a_{U,V,W}$ (left arrow), $a_{W,U,V}$ (right arrow), $\text{id}_U \otimes c_{V,W}$ (bottom-left arrow), $c_{U,W} \otimes \text{id}_V$ (bottom-right arrow)

It is easy to check that Rib_I^{NS} is a (non-strict) braided monoidal category (Rib_I^{NS} is only slightly more elaborate than Rib_I).

4.4 Balanced categories

In this section we define categories with twisting (inspired by ribbon graphs as in figure (4.2)). The definition is identical in both the strict and non-strict cases.

Definition 4.26. A (strict) **balanced category** is a braided (strict) monoidal category equipped with a family of natural isomorphisms (twists):

$$\{\theta_U : U \rightarrow U\} \quad (4.27)$$

such that the following *balancing diagram* commutes:

$$\begin{array}{ccc}
 U \otimes V & \xrightarrow{\theta_{U \otimes V}} & U \otimes V \\
 \theta_U \otimes \theta_V \downarrow & & \uparrow c_{V,U} \\
 U \otimes V & \xrightarrow{c_{U,V}} & V \otimes U
 \end{array} \quad (4.28)$$

This can be written as a formula for convenience:

$$\theta_{U \otimes V} = c_{V \otimes U} \circ c_{U \otimes V} \circ (\theta_U \otimes \theta_V) \quad (4.29)$$

Since the inspiration for this construction comes from ribbon graphs it is not surprising that Rib_I is a strict balanced category, and similarly Rib_I^{NS} is a (non-strict) balanced category. The balancing condition has a simple geometric interpretation in Rib_I - it is highly recommended for the reader to draw this out independently.

4.5 Right-Rigid monoidal categories

It is possible to rewind the discussion back to monoidal categories and consider a separate line of development (independent of braided monoidal and balanced categories). In this section we define a notion of duality. This is meant to

mimic duality in the category of vector spaces, however we note that there are many aspects of vector spaces that do not necessarily have analogues in this more general theory (for example there is no canonical isomorphism $V \rightarrow V^{**}$).

2

Right-rigid strict monoidal categories

Definition 4.30. A **right-rigid strict monoidal category** \mathcal{V} is a strict monoidal category such that for each object $V \in \text{Ob}(\mathcal{V})$ there is a distinguished **right dual** object V^* and morphisms (*not necessarily isomorphisms*)

$$b_V : \mathbb{1} \rightarrow V \otimes V^* \tag{4.31}$$

$$d_V : V^* \otimes V \rightarrow \mathbb{1}$$

These are *birth* and *death* morphisms. In addition we require that the following maps must be equal to id_V and id_{V^*} , respectively:

$$\begin{aligned} V &\xrightarrow{b_V \otimes \text{id}_V} V \otimes V^* \otimes V \xrightarrow{\text{id}_V \otimes d_V} V \\ V^* &\xrightarrow{\text{id}_{V^*} \otimes b_V} V^* \otimes V \otimes V^* \xrightarrow{d_V \otimes \text{id}_{V^*}} V^* \end{aligned} \tag{4.32}$$

²The connoisseur might be interested in following this branch further. Left duals can be defined similarly to right duals, and a right-left rigid monoidal category is simply called a *rigid monoidal category*. A *tensor category* has the simultaneous structure of a rigid monoidal category and an abelian category that has been enriched over finite-dimensional vector spaces (i.e. the Hom spaces are better than abelian groups - they are finite-dimensional \mathbb{C} -vector spaces; any characteristic 0 field k can be substituted for \mathbb{C}). The abelian structure and the monoidal structure must be compatible in the sense that \otimes distributes over \oplus . In addition we require $\text{Hom}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$.

A *finite tensor category* is a tensor category such that there are finitely-many *simple objects* (see below), each object can be decomposed as a finite-length list of simple objects, and each simple object admits a projective cover. If a finite tensor category is semisimple (stronger than the projective cover condition) then the category is a *fusion category*.

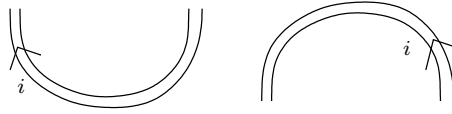


Figure 4.5: The birth $b_i : [] \rightarrow [[i, +1], [i, -1]]$ and death $d_i : [[i, -1], [i, +1]] \rightarrow []$ morphisms for the color i in the category Rib_I .

Rib_I is a right-rigid strict monoidal category. For a given object $[[i_1, \pm 1], [i_2, \pm 1], \dots, [i_k, \pm 1]]$ the dual object is $[[i_1, \mp 1], [i_2, \mp 1], \dots, [i_k, \mp 1]]$ (every $+1$ is changed to a -1 and vice versa). The birth and death morphisms are depicted in figure (4.5). The conditions in equation (4.32) have simple geometric interpretations in Rib_I and again it is in the interest of the reader to sketch these out.

(Non-strict) right-rigid monoidal categories

Definition 4.33. A **right-rigid monoidal category** \mathcal{V} is a monoidal category such that for each object $V \in \text{Ob}(\mathcal{V})$ there is a distinguished **right dual** object V^* and morphisms (*not necessarily isomorphisms*)

$$b_V : \mathbb{1} \rightarrow V \otimes V^* \tag{4.34}$$

$$d_V : V^* \otimes V \rightarrow \mathbb{1}$$

These are *birth* and *death* morphisms. Similar to the conditions above we require that the following maps must be equal to id_V and id_{V^*} , respectively:

$$\begin{aligned}
V &\xrightarrow{l_V^{-1}} \mathbb{1} \otimes V \xrightarrow{b_V \otimes \text{id}_V} (V \otimes V^*) \otimes V \xrightarrow{a_{V, V^*, V}} \\
&V \otimes (V^* \otimes V) \xrightarrow{\text{id}_V \otimes d_V} V \otimes \mathbb{1} \xrightarrow{r_V} V \\
V^* &\xrightarrow{r_{V^*}^{-1}} V^* \otimes \mathbb{1} \xrightarrow{\text{id}_{V^*} \otimes b_V} V^* \otimes (V \otimes V^*) \xrightarrow{a_{V^*, V, V^*}^{-1}} \\
&(V^* \otimes V) \otimes V^* \xrightarrow{d_V \otimes \text{id}_{V^*}} \mathbb{1} \otimes V^* \xrightarrow{l_{V^*}} V^*
\end{aligned} \tag{4.35}$$

The only difference is that the associativity maps appear.

In a similar fashion to Rib_I it is easy to show that Rib_I^{NS} is a (non-strict) right-rigid monoidal category.

4.6 Ribbon categories

Ribbon categories were studied in [Shu94]. The definitions for strict and non-strict ribbon categories are nearly identical, hence we define them simultaneously.

Definition 4.36. A (strict) ribbon category is a right-rigid (strict) monoidal category that in addition is a (strict) balanced category.

The balancing and rigidity must be compatible:

$$(\theta_V \otimes \text{id}_{V^*}) \circ b_V = (\text{id}_V \otimes \theta_{V^*}) \circ b_V \tag{4.37}$$

(again the geometric picture in Rib_I is illuminating).

We now describe some properties of ribbon categories. First, given an

object V in a ribbon category \mathcal{V} and a morphism $f : V \rightarrow V$ we define the **quantum trace** of f :

$$\mathrm{tr}_q(f : V \rightarrow V) := d_V \circ c_{V,V^*} \circ ((\theta_V \circ f) \otimes \mathrm{id}_{V^*}) \circ b_V \quad (4.38)$$

Furthermore the **quantum dimension** is defined by:

$$\mathrm{dim}_q(V) := \mathrm{tr}_q(\mathrm{id}_V) = d_V \circ c_{V,V^*} \circ (\theta_V \otimes \mathrm{id}_{V^*}) \circ b_V \quad (4.39)$$

We note that if the objects in the underlying category have some underlying *intrinsic* notion of trace and dimension (e.g. the objects are finite-dimensional vector spaces) then it is *not true* that the quantum trace and quantum dimension necessarily agree with the intrinsic notions. For example the quantum dimension need *not* even be an integer.

Every ribbon category is **pivotal**, that is for each object V there is a distinguished isomorphism $V \xrightarrow{\sim} V^{**}$ determined by the composition: ³

$$\begin{array}{ccc} V \xrightarrow{\mathrm{id}_V \otimes b_{V^*}} V \otimes V^* \otimes (V^*)^* & \xrightarrow{\theta_V \otimes \mathrm{id}_{V^*} \otimes \mathrm{id}_{V^{**}}} & \\ \\ V \otimes V^* \otimes (V^*)^* & \xrightarrow{c_{V,V^*} \otimes \mathrm{id}_{V^{**}}} V^* \otimes V \otimes (V^*)^* & \xrightarrow{d_V \otimes \mathrm{id}_{V^{**}}} V^{**} \end{array} \quad (4.40)$$

Again (if the objects are finite-dimensional vector spaces) this isomorphism is typically *not* the same as the canonical vector space isomorphism $V \xrightarrow{\sim} V^{**}$.

It is also a fact that ribbon categories are **spherical**, that is $\mathrm{dim}_q(V) =$

³This composition makes sense for strict ribbon categories. There is a similar composition for non-strict ribbon categories. We note that it is *not* obvious that this composition of morphisms is an *isomorphism*. This can be proven using the functor F introduced in the next section (see [Tur94] pg. 40).

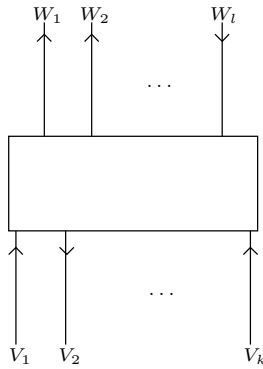


Figure 4.6: An elementary (k, l) -ribbon graph.

$\dim_q(V^*)$ for every object. The proof requires the functor F discussed in the next section.

4.7 Invariants of colored (k, l) -ribbon graphs using ribbon categories

In the last several sections we have been considering the category Rib_I where I is some arbitrary labeling set. Suppose that we replace I with a right-rigid strict monoidal category \mathcal{V} and consider $\text{Rib}_{\mathcal{V}}$, i.e. we color the oriented ribbons (and annuli) with *objects* in \mathcal{V} . Because of the right-rigid strict monoidal structure we can go further and color the *coupons* with *morphisms* as well. We discuss this now.

First consider an *elementary* (k, l) -ribbon graph in standard drawing position as depicted in figure (4.6). The graph is called “elementary” because there is neither braiding nor twisting in any of the ribbons (neither birth nor death), there is a single coupon, and all of the ribbons terminate on the coupon.

Denote $V^{+1} := V$ and $V^{-1} := V^*$. Then it makes sense to color the coupon in figure (4.6) with a morphism

$$\phi \in \text{Hom}(V_1^{\pm 1} \otimes \dots \otimes V_k^{\pm 1}, W_1^{\pm 1} \otimes \dots \otimes W_l^{\pm 1}) \quad (4.41)$$

where we use $+1$ for ribbons pointing “up” and -1 for ribbons pointing “down”. Note that both the monoidal and rigidity properties of \mathcal{V} have been used. In this way we can color coupons in any arbitrary colored (k, l) -ribbon graph.

Let us introduce the terminology **fully colored (k, l) -ribbon graphs** for ribbon graphs where all of the coupons are colored with morphisms. Using this enrich $\text{Rib}_{\mathcal{V}}$ by replacing the morphisms (colored (k, l) -ribbon graphs) with *fully colored (k, l) -ribbon graphs*.

Generalizing the above construction to the non-strict case $\text{Rib}_{\mathcal{V}}^{\text{NS}}$ is straightforward and left to the reader.

The main functor F

We can go further and consider $\text{Rib}_{\mathcal{V}}$ where \mathcal{V} is now a *strict ribbon category*. Then we have two strict ribbon categories to consider: $\text{Rib}_{\mathcal{I}}$ (which is a strict ribbon category since any Rib_I is) and \mathcal{V} . The main theorem for ribbon categories is the following (proven by Reshetikhin and Turaev in the language of quantum groups):

Theorem 4.42 (Reshetikhin, Turaev). *Let \mathcal{V} be a **strict** ribbon category. Consider the enriched **strict** ribbon category $\text{Rib}_{\mathcal{V}}$ (enriched means the morphisms are fully colored (k, l) -ribbon graphs). Set notation for primitive ribbon*

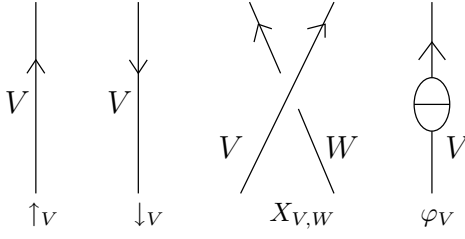


Figure 4.7: A list of primitive ribbons graphs. The notation will be used in theorem (4.42)

graphs as in figure (4.7). Then there is a unique **strict** monoidal functor

$$F : \text{Rib}_{\mathcal{V}} \rightarrow \mathcal{V} \tag{4.43}$$

such that

$$F([[V, +1]]) = V \tag{4.44}$$

$$F([[V, -1]]) = V^*$$

$$F(\uparrow_V) = id_V$$

$$F(\downarrow_V) = id_{V^*}$$

$$F(X_{V,W}) = c_{V,W}$$

$$F(\varphi_V) = \theta_V$$

We have not seen a non-strict version of this theorem stated and proven in the literature, however the following analogue is almost certainly true (and we implicitly use it in the remainder of this paper):

Conjecture 4.45. *Let \mathcal{V} be a ribbon category. Consider the enriched ribbon category $\text{Rib}_{\mathcal{V}}^{NS}$ (enriched means the morphisms are fully colored non-associative (k, l) -ribbon graphs). Set notation for primitive ribbon graphs as in*

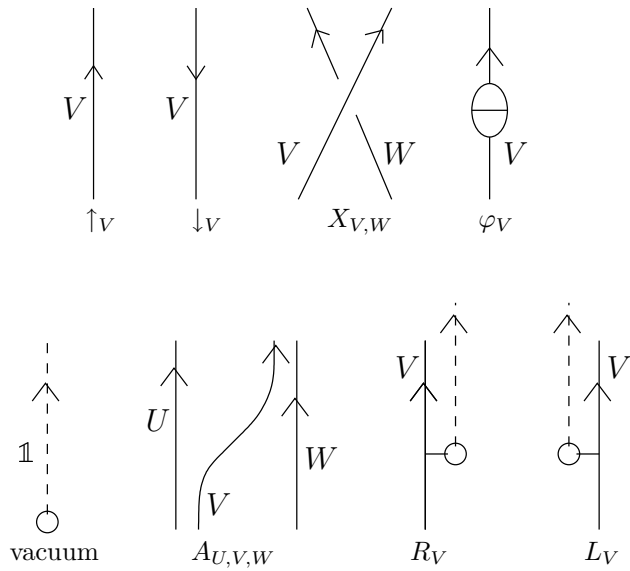


Figure 4.8: A list of primitive “non-associative” ribbons graphs. The notation will be used in conjecture (4.45)

figure (4.8). Then there is a unique monoidal functor

$$F : Rib_{\mathcal{V}}^{NS} \rightarrow \mathcal{V} \tag{4.46}$$

such that

$$F([[V, +1]]) = V \tag{4.47}$$

$$F([[V, -1]]) = V^*$$

$$F(\uparrow_V) = id_V$$

$$F(\downarrow_V) = id_{V^*}$$

$$F(X_{V,W}) = c_{V,W}$$

$$F(\varphi_V) = \theta_V$$

$$F(A_{U,V,W}) = a_{U,V,W}$$

$$F(R_V) = r_V$$

$$F(L_V) = l_V$$

4.8 Modular tensor categories

In this section we define modular tensor categories. We shall make no reference to strict/non-strict categories, leaving it to the reader to make the appropriate substitutions where necessary.

We begin with a preliminary definition:

Definition 4.48. Consider a preadditive category \mathcal{V} that also is enriched so that the Hom sets are \mathbb{C} -vector spaces (rather than just abelian groups). Then a **simple object** V_x is an object such that

$$\text{Hom}(V_x, V_x) \cong \mathbb{C} \tag{4.49}$$

Suppose that \mathcal{V} is an enriched preadditive category and in addition is

a ribbon category. We require that the preadditive structure be compatible with the monoidal structure (i.e. \otimes distributes over $+$ of morphisms). Then it is straightforward to check that the dual V_x^* of a simple object is also simple. It is also straightforward to prove that $\mathbb{1}$ is a simple object.

The definition of a modular tensor category in [Tur94] is based on preadditive ribbon categories and is slightly more general than what is presented below. We restrict attention to *additive* ribbon categories:

Definition 4.50. A **modular tensor category** is a category with the following structure:

1. Ribbon category
2. Additive category enriched over \mathbb{C} -vector spaces
3. Ribbon/additive compatibility (\otimes distributes over \oplus)
4. Semisimple with *finitely*-many simple objects
5. The **S -matrix** is invertible, where S is defined by (using the ribbon structure on simple objects V_x and V_y):

$$S_{x,y} := \text{tr}_q(c_{V_y, V_x^*} \circ c_{V_x^*, V_y}) \quad (4.51)$$

6. A choice of square root

$$\mathcal{D} := \sqrt{\sum_{\text{simple objects}} (\dim_q(V_x))^2} \quad (4.52)$$

Since for a simple object $\text{Hom}(V_x, V_x) \cong \mathbb{C}$ we see that the twist isomorphism $\theta_{V_x} : V_x \rightarrow V_x$ is given by a complex number (denoted θ_x).

The following expressions will be used often in the sequel:

$$\begin{aligned}
p_+ &:= \sum_{\text{simple objects}} (\dim_q(V_x))^2 \theta_x & (4.53) \\
p_- &:= \sum_{\text{simple objects}} (\dim_q(V_x))^2 \theta_x^{-1}
\end{aligned}$$

It is a fact (see [BK00]) that

$$\mathcal{D}^2 = p_+ p_- \tag{4.54}$$

4.9 Invariants of 3-manifolds, 2+1-dimensional TQFTs from MTCs

We mentioned in section (4.7) that associated to any ribbon category \mathcal{V} is a monoidal functor

$$F : \text{Rib}_{\mathcal{V}} \rightarrow \mathcal{V} \tag{4.55}$$

Using this functor it is straightforward to assign to any fully-colored (k, l) -ribbon graph in \mathbb{R}^3 a morphism $V_1^{\pm 1} \otimes \dots \otimes V_k^{\pm 1} \rightarrow W_1^{\pm 1} \otimes \dots \otimes W_l^{\pm 1}$ between the object coloring the bottom of the graph and the object coloring the top. It is proven in [Tur94] that the resulting morphism is invariant under regular isotopy of the ribbon graph.

Now we turn our attention to modular tensor categories. We shall see that the stronger structure allows us to define invariants of closed oriented 3-manifolds (and, eventually, 2+1 TQFTs). Before we begin suppose first that we have a ribbon graph in S^3 . It is easy to isotope any ribbon graph in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ appropriately to “miss” the point $\{\infty\}$, hence we can consider

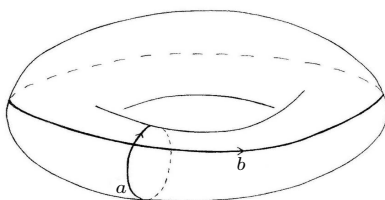


Figure 4.9: A torus with oriented meridian and longitude.

the ribbon graph as embedded in \mathbb{R}^3 (where we can apply the functor F).

Since we wish to study closed oriented 3-manifolds X the following standard theorem is useful: ⁴

Theorem 4.56 (Dehn, Lickorish). *Any orientable closed 3-manifold X can be obtained from S^3 by drilling out solid tori and gluing them back in along different diffeomorphisms (up to isotopy) of their boundaries. Furthermore, each such surgery can be assumed to be an “integer surgery” (see below).*

Surgery

The diffeomorphisms along which we reglue the solid tori can be neatly encoded in terms of framed links in S^3 . This can be seen by considering each solid torus individually. Before drilling out the solid torus pick a reference longitude b and meridian a on the boundary as in figure (4.9).

From chapter (2) we know that $\text{MCG}(T^2) \cong \text{PSL}(2, \mathbb{Z})$. In particular a

⁴Actually the original theorem requires *rational* surgery, but there is a well-known algorithm to reduce from rational surgery to integer surgery (see, e.g., [PS96]). Since we will not require rational surgery we do not bother here.

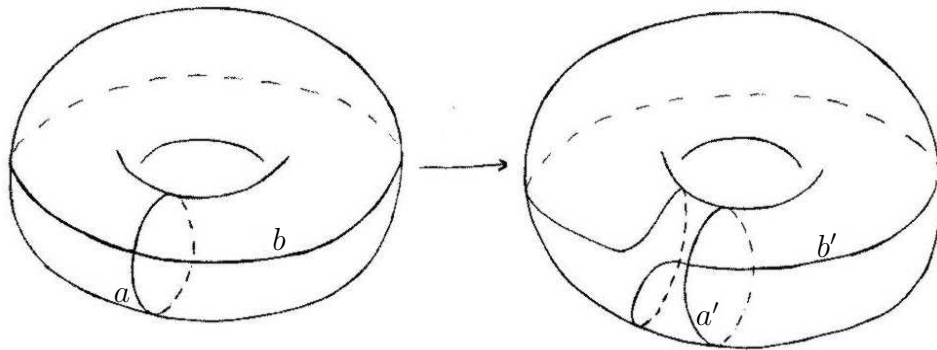


Figure 4.10: A Dehn twist on the curve a .

diffeomorphism is determined by the action on homology generators ⁵

$$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (4.57)$$

Consider the effect of drilling out a single torus and gluing it back in along the diffeomorphism determined by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (4.58)$$

This is depicted in figure (4.10). It is not difficult to convince oneself that this surgery does *not change* the topology of the 3-manifold (removing a solid torus, cutting it, twisting it, gluing it together, and replacing it in the hole is the same as simply filling in the hole). More generally the surgery determined

⁵This is not true in higher genus.

by the boundary diffeomorphism

$$T^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \tag{4.59}$$

also does not change the topology of the 3-manifold.

Because of this observation we have the following common fact (we could not find the simple argument written down, hence we write it here for completeness):

Fact 4.60. *A surgery on a single solid torus is determined by specifying two relatively-prime integers q and p . We say that the ratio $\frac{q}{p}$ determines a **ratio-nal** surgery. In fact we only have to specify the image of a*

$$a \mapsto q \cdot a + p \cdot b \tag{4.61}$$

Proof. We construct a matrix

$$\begin{pmatrix} q & -r \\ p & s \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \tag{4.62}$$

for some integers r and s . Since the determinant must be 1, we want to find integers r and s such that

$$qs + pr = 1 \tag{4.63}$$

However since q and p are relatively prime the Euclidean algorithm can be used to find suitable integers r and s that satisfy the above equation. The choice is not unique since $r - kq$ and $s + kp$ also works for any integer k .

We need to know how the surgeries determined by the matrices

$$\begin{pmatrix} q & -r \\ p & s \end{pmatrix} \quad \begin{pmatrix} q & -(r - kq) \\ p & s + kp \end{pmatrix} \quad (4.64)$$

differ. It is easy to check that

$$\begin{pmatrix} q & -(r - kq) \\ p & s + kp \end{pmatrix} = \begin{pmatrix} q & -r + kq \\ p & s + kp \end{pmatrix} = \begin{pmatrix} q & -r \\ p & s \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q & -r \\ p & s \end{pmatrix} T^k \quad (4.65)$$

Hence the surgeries differ by precomposing with a T^k surgery (which we already argued does not change the topology of the 3-manifold).

This proves that a surgery along a single solid torus is determined by two relatively prime integers q and p . \square

When $p = 1$ this is *integer* surgery. There is a standard algorithm that reduces rational surgery to integer surgery (by continued fraction expansion and drilling out more solid tori, see [PS96]) hence we set $p = 1$ from now on. Therefore a surgery along a single solid torus is determined by a single integer q and we have the following corollary:

Corollary 4.66. *Any closed oriented 3-manifold X can be presented as a surgery along **framed** links in S^3 .*

Proof. Dehn-Lickorish implies that any closed oriented 3-manifold X can be obtained by drilling out/regluing solid tori in S^3 . If we consider the cores of the tori this determines a link in S^3 (from the link components we could recover the solid tori by thickening). The only issue is how to encode the regluing diffeomorphism. We have seen that any integer surgery (along a

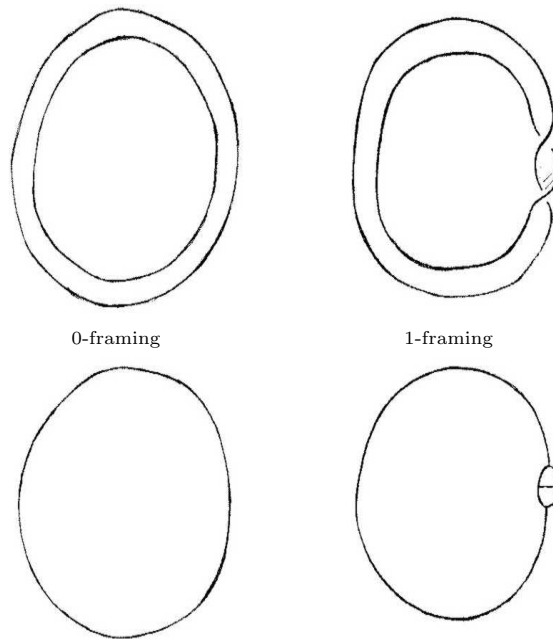


Figure 4.11: Framed unknots with 0 and 1 framing, respectively. The lower diagrams are the skeletal schematic diagrams.

single solid torus) is determined by a single integer q , hence we can frame the corresponding link component with the appropriate framing number q . Repeating this for all of the solid tori produces a framed link in S^3 that determines the surgery completely. \square

Example 4.67. The most important example is the *torus switch*, i.e. surgery along a framed *unknot* with framing number 0 (see the left side of figure (4.11)).

Since $q = 0$ we have that the following matrix determines the surgery

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.68}$$

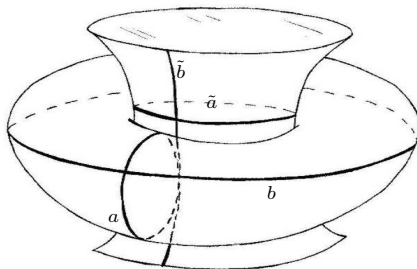


Figure 4.12: Heegaard decomposition of S^3 into two solid tori. The “plug” is a solid torus that has been cut. Imagine deforming the plug (as shown) and enveloping completely the other solid torus to form a 3-ball with boundary S^2 (i.e. identify the longitude of the solid torus with the meridian of the plug, and the meridian of the solid torus with the longitude of the plug). Since the plug is actually a cut solid torus we know that the top hemisphere of the boundary S^2 should be identified with the bottom hemisphere. Topologically this is not different than crushing the entire S^2 to a point. However, the 3-ball with boundary S^2 crushed to a point is just S^3 . We do not draw the orientations for a , b , \tilde{a} , and \tilde{b} , however a quick check verifies that $a \leftrightarrow \tilde{b}$ and $b \leftrightarrow \tilde{a}$, which is an orientation *reversing* gluing diffeomorphism as expected since we can only glue outgoing boundaries to incoming boundaries.

This sends

$$a \mapsto b \quad b \mapsto -a \tag{4.69}$$

i.e. the longitude and meridian swap roles (this is an orientation preserving map).

On the other hand consider the Heegaard decomposition of S^3 into two solid tori depicted in figure (4.12). We have the identifications

$$\begin{aligned} a &\leftrightarrow \tilde{b} \\ b &\leftrightarrow \tilde{a} \end{aligned} \tag{4.70}$$

If we drill out one of the tori from figure (4.12), apply the self-diffeomorphism

determined by the S matrix given above, and reglue then we have the identifications ⁶

$$a \mapsto b \Rightarrow b \leftrightarrow \tilde{b} \tag{4.71}$$

$$b \mapsto -a \Rightarrow -a \leftrightarrow \tilde{a}$$

In other words we have two solid tori that are glued together (longitude to longitude, meridian to meridian). Since a solid torus is just $D^2 \times S^1$ where the S^1 factor can be identified with b , and the boundary of the disk D^2 can be identified with a , we see that gluing the two solid tori together gives $S^2 \times S^1$ (for a fixed point on the longitude S^1 both solid tori look like $D^2 \times \{\text{pt}\}$ - gluing two disks together along the boundary gives us a 2-sphere $S^2 \times \{\text{pt}\}$).

Summarizing, a surgery along a 0-framed unknot in S^3 gives the closed oriented 3-manifold $S^2 \times S^1$. Iterating the surgery again we recover S^3 .

Example 4.72. It is shown in [PS96] that a surgery with framing number ± 1 (see the right side of figure (4.11)) along an isolated unknot is trivial, i.e. the 3-manifold topology does not change. For example the diffeomorphism for the +1 framing is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{4.73}$$

and the proof that this does not alter the topology of the 3-manifold is similar

⁶We need to be careful with orientations, i.e. the S matrix is an orientation preserving self-diffeomorphism, but the cutting and regluing are orientation reversing operations.

to the proof that the T matrix diffeomorphism

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{4.74}$$

does not alter the 3-manifold. We note that this only applies to isolated unknots. For contrast ± 1 -framed surgery along a component that is *linked* is *nontrivial*.

In general, given an oriented closed 3-manifold presented by some other means (say, a Heegaard decomposition), it may be difficult to provide a surgery presentation of framed links in S^3 . Furthermore, the surgery presentation is certainly not unique (considering the example above, we could add as many ± 1 -framed isolated unknots to the diagram as desired and not change the resulting 3-manifold. Likewise we could add even pairs of torus switches along isolated unknots and still not change the resulting 3-manifold).

However, any two surgery presentations of the same 3-manifold can be related by the **Kirby moves** (see, e.g., [PS96]). Since we do not require these moves explicitly (and since they are standard) we omit their description. However, we note that the proof that a modular tensor category gives 3-manifold invariants essentially reduces to showing invariance under the Kirby moves.

Invariants of closed 3-manifolds from MTCs

Once a surgery presentation is specified for X the computation of the 3-manifold invariant is straightforward. The strategy is to *average* over all

possible colorings of the framed link L in S^3 .⁷ We pick an orientation on each of the components of $L = \{L_1, \dots, L_m\}$. The chosen orientation does not affect the invariant because we are summing over *all* colorings.⁸

Note that in general we may allow the 3-manifold X to also contain some embedded oriented *fixed colored* ribbon graph Ω in addition to the oriented framed link L . It is understood that Ω does not participate in the surgery.

If we pick a coloring for L by **simple** objects $\{V_i\}_{i \in I}$ then we can compute the ribbon graph invariant $F(L \cup \Omega)$. Denote by V_{λ_i} the coloring of the link component L_i .

We require a normalization convention. Every *oriented* framed link $L = \{L_1, \dots, L_m\}$ has an $m \times m$ linking number matrix B where an off-diagonal element is given by

$$B_{ij} = \text{lk}(L_i, L_j) = \frac{\# \text{ positive crossings} - \# \text{ negative crossings}}{2} \quad (4.75)$$

and a diagonal element is just

$$B_{ii} = \text{framing number of } L_i \quad (4.76)$$

Denote the signature of this matrix by $\sigma(L)$. Then, given a surgery presentation for X as a framed link L in S^3 we compute the 3-manifold invariant

$$\tau(X) := p_-^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_{\text{col of } L} \left(\prod_{i=1}^m \dim_q(V_{\lambda_i}) \right) F(L \cup \Omega) \quad (4.77)$$

The components of L can only be colored by simple objects.

⁷Hence the necessity for finitely-many simple objects.

⁸Recall that we can switch orientation if we replace a coloring V with the dual V^* .

$(2 + 1)$ -dimensional topological quantum field theory

The 3-manifold invariant provided in the last subsection can be exploited further to produce an extended $(2 + 1)$ -dimensional TQFT in the sense of chapter (2). Consider an oriented 3-manifold X with boundary $\partial X = -\Sigma_- \sqcup \Sigma_+$. For simplicity we assume that Σ_- and Σ_+ are connected *closed* 2 surfaces ⁹

It was stated in chapter (2) that an *extended* structure is required on 2-surfaces and 3-bordisms in order to define an anomaly-free $(2 + 1)$ -dimensional TQFT. However for *closed* 3-manifolds there is a canonical choice for this extended structure (see [Ati90b]) and hence we did not need to mention it in the previous subsection concerning 3-manifold invariants.

We now place a strong structure on a boundary 2-surface Σ . We say that Σ is **parameterized** if it is equipped with a fixed diffeomorphism

$$\phi : \partial H_g \rightarrow \Sigma \tag{4.78}$$

where H_g is a **standard** handlebody that we now specify. ¹⁰

Standard handlebodies

We define the standard handlebody H_g of genus g as a thickening of the standard uncolored ribbon graph R_g (embedded in \mathbb{R}^3) depicted in figure (4.13). The boundary is a surface Σ_g of genus g . The handlebody H_g inherits an

⁹In general colored ribbon graphs Ω can terminate on the boundary forming *marked arcs*. As mentioned in chapter (2) these can also be viewed as parameterized boundary circles (from the perspective of conformal field theory).

¹⁰In the same spirit as the Segal modular functor (where the complex structure turned out to be irrelevant when defining a projective representation of $\text{MCG}(\Sigma)$) the parameterization is irrelevant if we are content with TQFTs with gluing anomaly, and the dependence is weak for a full anomaly-free TQFT.

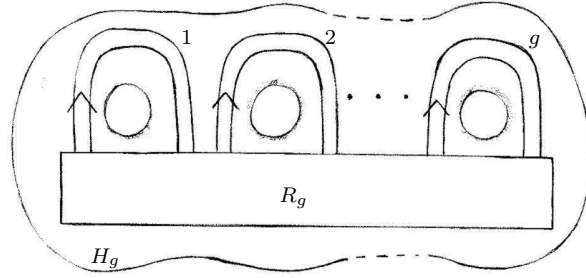


Figure 4.13: Standard handlebody H_g with the standard embedded ribbon graph R_g .

orientation from its embedding in \mathbb{R}^3 . We endow Σ_g with the orientation that agrees with the boundary orientation, i.e. $\Sigma_g = \partial H_g$. In this sense Σ_g is *outgoing*.

Likewise we define the standard handlebody \overline{H}_g as a thickening of the standard uncolored ribbon graph \overline{R}_g depicted in figure (4.14).¹¹ Again the boundary is a surface $\overline{\Sigma}_g$ of genus g . \overline{H}_g inherits an orientation from its embedding in \mathbb{R}^3 . However here we supply $\overline{\Sigma}_g$ with the opposite orientation from the boundary orientation, i.e. $\overline{\Sigma}_g = -\partial \overline{H}_g$. In this sense $\overline{\Sigma}_g$ is *incoming*. There is a natural identification¹²

$$\overline{\Sigma}_g = -\Sigma_g \tag{4.79}$$

We can color the ribbons of R_g (in order from left to right) with simple objects $\{V_{\lambda_1}, \dots, V_{\lambda_g}\}$. Denote the resulting ribbon graph by

$$R_g(V_{\lambda_1}, \dots, V_{\lambda_g}) \tag{4.80}$$

¹¹Note that \overline{R}_g is not exactly a mirror image of R_g .

¹²The construction is more complicated in the presence of marked arcs.

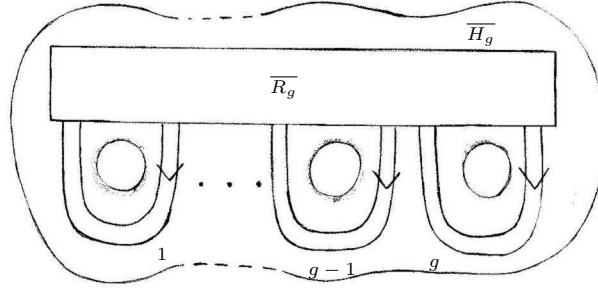


Figure 4.14: Standard handlebody \overline{H}_g with the standard embedded ribbon graph \overline{R}_g .

We can also then color the *coupon* with a morphism

$$f : \mathbb{1} \rightarrow V_{\lambda_1} \otimes V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_g} \otimes V_{\lambda_g}^* \quad (4.81)$$

Denote the resulting fully-colored ribbon graph by

$$R_g(V_{\lambda_1}, \dots, V_{\lambda_g}; f) \quad (4.82)$$

Similarly we can color the ribbons of \overline{R}_g with simple objects $\{V_{\zeta_1}, \dots, V_{\zeta_g}\}$ and denote the resulting ribbon graph

$$\overline{R}_g(V_{\zeta_1}, \dots, V_{\zeta_g}) \quad (4.83)$$

Likewise can then color the *coupon* with a morphism

$$h : V_{\zeta_1} \otimes V_{\zeta_1}^* \otimes \cdots \otimes V_{\zeta_g} \otimes V_{\zeta_g}^* \rightarrow \mathbb{1} \quad (4.84)$$

Denote the fully colored ribbon graph by

$$\overline{R}_g(V_{\zeta_1}, \dots, V_{\zeta_g}; h) \quad (4.85)$$

If we color the ribbons of R_g and \overline{R}_g with the *same* ordered list of simple objects $\{V_{\lambda_1}, \dots, V_{\lambda_g}\}$ then we have

$$f \in \text{Hom}(\mathbb{1}, V_{\lambda_1} \otimes V_{\lambda_1}^* \otimes \dots \otimes V_{\lambda_g} \otimes V_{\lambda_g}^*) \quad (4.86)$$

$$h \in \text{Hom}(V_{\lambda_1} \otimes V_{\lambda_1}^* \otimes \dots \otimes V_{\lambda_g} \otimes V_{\lambda_g}^*, \mathbb{1}) = \quad (4.87)$$

$$= (\text{Hom}(\mathbb{1}, V_{\lambda_1} \otimes V_{\lambda_1}^* \otimes \dots \otimes V_{\lambda_g} \otimes V_{\lambda_g}^*))^* \quad (4.88)$$

where the last line follows from the natural pairing

$$h(f) := h \circ f \in \text{Hom}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C} \quad (4.89)$$

In this way we see that a colored coupon in \overline{R}_g lives in the dual space of a colored coupon in R_g .

Hilbert space of states

Now we describe how to associate a vector space (or Hilbert space if the theory is unitary - see [Tur94]) to an oriented closed surface Σ of genus g equipped with a parameterization $\phi : \partial H_g \rightarrow \Sigma$. Again this construction can be straightforwardly generalized to surfaces with marked arcs.

Since the surface is parameterized we identify it as the boundary of the standard handlebody H_g . The embedded ribbon R_g is uncolored. The idea is

to sum over all possible colorings of R_g . Define the associated vector space:

$$\mathcal{F}(\Sigma) := \bigoplus_{\text{col } \{V_{\lambda_1}, \dots, V_{\lambda_g}\}} \text{Hom}(\mathbb{1}, V_{\lambda_1} \otimes V_{\lambda_1}^* \otimes \dots \otimes V_{\lambda_g} \otimes V_{\lambda_g}^*) \quad (4.90)$$

This defines part of the non-extended modular functor from chapter (2)). It is still necessary to describe the action of diffeomorphisms $\Sigma \rightarrow \Sigma$ on $\mathcal{F}(\Sigma)$.

Operators associated to oriented 3-bordisms

Recall that X is an oriented 3-manifold with boundary $\partial X = \partial X_- \sqcup \partial X_+ = -\Sigma_- \sqcup \Sigma_+$. For simplicity we assume that Σ_- and Σ_+ are connected *closed* 2 surfaces of genus g_- and g_+ , respectively. In addition assume that we have an extended structure on the boundaries, i.e. parameterizations

$$\begin{aligned} \phi_- : \partial H_{g_-} &\rightarrow \Sigma_- \\ \phi_+ : \partial H_{g_+} &\rightarrow \Sigma_+ \end{aligned} \quad (4.91)$$

From the axioms for a TQFT we expect to assign to the 3-bordism X an operator

$$\tau(X) : \mathcal{F}(\Sigma_-) \rightarrow \mathcal{F}(\Sigma_+) \quad (4.92)$$

The matrix elements of the operator $\tau(X)$ are determined by the following recipe: pick a basis for $\mathcal{F}(\Sigma_-)$ (and for $\mathcal{F}(\Sigma_+)$). The vector space $\mathcal{F}(\Sigma_-)$ is defined by equation (4.90) (and similarly for $\mathcal{F}(\Sigma_+)$). Using the parameterizations of Σ_- and Σ_+ we “cap off” X with the standard handlebodies to produce a *closed* 3-manifold \tilde{X} (with embedded *uncolored* ribbon graphs R_{g_-} $\overline{R_{g_+}}$ in the handlebodies).

Choosing a specific basis element out of $\mathcal{F}(\Sigma_-)$ and a specific basis element out of $\mathcal{F}(\Sigma_+)$ is the same as specifying a coloring for R_{g_-} and $\overline{R_{g_+}}$. Then calculating the 3-manifold invariant $\tau(\tilde{X}) \in \mathbb{C}$ gives the corresponding matrix element for the operator

$$\tau(X) : \mathcal{F}(\Sigma_-) \rightarrow \mathcal{F}(\Sigma_+) \quad (4.93)$$

We only need to be careful about orientations to ensure that the correct handlebodies are glued onto the correct boundary components. Recall that in order to maintain an overall well-defined orientation under gluing it is necessary to stipulate that incoming boundary components can only be glued to outgoing boundary components (i.e. the gluing diffeomorphisms must be orientation *reversing*).

Since the defined orientation of Σ_- disagrees with its induced orientation as part of the boundary ∂X_- we can use the parameterization $\phi_- : \partial H_{g_-} \rightarrow \Sigma_- = -\partial X_-$ to glue H_{g_-} to X . This is an orientation-reversing diffeomorphism, and effectively caps off Σ_- .

Now consider Σ_+ . Here the defined orientation agrees with the boundary orientation ∂X_+ , so we cannot glue using the parameterization $\phi_+ : \partial H_{g_+} \rightarrow \Sigma_+ = \partial X_+$ since this is orientation preserving. However, we can use the mirror standard handlebody $\overline{H_{g_+}}$ instead since we have a natural identification $-\partial \overline{H_{g_+}} = \partial H_{g_+}$. The same map ϕ_+ is orientation *reversing*

$$\phi_+ : \partial \overline{H_{g_+}} \rightarrow \Sigma_+ = \partial X_+ \quad (4.94)$$

so we cap off Σ_+ with the standard handlebody $\overline{H_{g_+}}$.

Mapping class group

In particular the operator assignment $\tau(X)$ for a 3-bordism X provides a (projective - see below) representation of the mapping class group for any surface Σ of genus g .

Consider the surface Σ parameterized by a fixed diffeomorphism $\phi : \partial H_g \rightarrow \Sigma$. Form the 3-manifold $X = \Sigma \times I$ with boundary $\partial X = -\Sigma \sqcup \Sigma$ where both the incoming and outgoing boundary components have the *same* parameterization ϕ . The operator associated to $\Sigma \times I$ (using the above procedure) is just the identity

$$\tau(X) = \text{id} : \mathcal{F}(\Sigma) \rightarrow \mathcal{F}(\Sigma) \quad (4.95)$$

Now assume the we have some isotopy class of diffeomorphisms $[f] \in \text{MCG}(\Sigma)$ that are *not* isotopic to the identity. Picking a representative diffeomorphism $f : \Sigma \rightarrow \Sigma$ we form a new 3-manifold $X_f = \Sigma \times I$ where the outgoing boundary component Σ is still parameterized by ϕ , however the incoming boundary component Σ is parameterized instead by the map $f \circ \phi$. Clearly when we “cap off” with standard handlebodies the resulting closed 3-manifold \tilde{X}_f will be different, hence the operator $\tau(X_f)$ will not be the identity, but instead a nontrivial operator

$$\tau(X_f) : \mathcal{F}(\Sigma) \rightarrow \mathcal{F}(\Sigma) \quad (4.96)$$

In this way we can associate to any element f of $\text{MCG}(\Sigma)$ a linear operator $\tau(X_f) : \mathcal{F}(\Sigma) \rightarrow \mathcal{F}(\Sigma)$. The composition of diffeomorphisms $g \circ f$ can be realized by gluing the outgoing boundary component of X_f to the

incoming boundary component of X_g , so we have

$$\tau(X_{g \circ f}) = \tau(X_g \cup_{\text{glued}} X_f) = k\tau(X_g) \circ \tau(X_f) \quad (4.97)$$

using the gluing properties outlined in chapter (2). Notice the gluing anomaly k , hence we have a *projective* representation of $\text{MCG}(\Sigma)$.

4.10 Trivial examples from (\mathcal{D}, q, c)

In chapter (3) it was shown that the quantum data for (non-spin) toral Chern-Simons theories is encoded in a finite abelian group \mathcal{D} , a *pure* quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$, and c (an integer mod 24) that encodes a choice of cube root of the Gauss reciprocity formula.

An easy semisimple ribbon category that can be formed (mentioned in the appendix of [Del99]) from (\mathcal{D}, q, c) is the group algebra $\mathbb{C}[\mathcal{D}]$ where we write \mathcal{D} multiplicatively:

1. For each $x \in \mathcal{D}$ we define a simple object $\mathbb{C}x$ (a copy of \mathbb{C} labeled by the group element). An arbitrary object is defined to be a formal \oplus of simple objects.
2. Define the tensor product using the group law, i.e. $\mathbb{C}x \otimes \mathbb{C}y = \mathbb{C}xy$ (extend to arbitrary objects using additivity).
3. A morphism $k : \mathbb{C}x \rightarrow \mathbb{C}x$ from a simple object to itself is just multiplication by a complex number k . The set $\text{Mor}(\mathbb{C}x, \mathbb{C}y)$ for $x \neq y$ contains only the zero morphism 0. Extend to arbitrary objects in the obvious way by additivity.

4. It is easy to check that $\mathbb{C}[\mathcal{D}]$ is a strict monoidal category.

We define a ribbon category $\mathbb{C}[\mathcal{D}]_{(\mathcal{D}, q, c)}$ (recall $b : \mathcal{D} \otimes \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ is the bilinear form induced from q):

1. The braiding $c_{x,y} : \mathbb{C}x \otimes \mathbb{C}y \rightarrow \mathbb{C}y \otimes \mathbb{C}x$ on two simple objects is defined as (since in this case $\mathbb{C}x \otimes \mathbb{C}y \cong \mathbb{C}y \otimes \mathbb{C}x \cong \mathbb{C}xy$)

$$\mathbb{C}xy \rightarrow \mathbb{C}xy \quad \text{multiplication by } \exp(2\pi i b(x, y)) \quad (4.98)$$

2. Enforcing the balancing condition (and using the fact that $b(x, y) = q(x + y) - q(x) - q(y)$ we see that the twist for a simple object is

$$\mathbb{C}x \rightarrow \mathbb{C}x \quad \text{multiplication by } \exp(2\pi i 2q(x)) \quad (4.99)$$

It is easy to compute the S -matrix:

$$S_{xy} = \exp(2\pi i 2b(-x, y)) = \exp(-2\pi i 2b(x, y)) \quad (4.100)$$

However it is easy to see that for $U(1)$ at level B where B is an even integer the above S -matrix is singular. For example (see chapter (3)), for $B = 2$ the discriminant group is $\mathcal{D} \cong \mathbb{Z}_2 = \{0, 1/2\}$, and the only non-degenerate bilinear form is determined by

$$b(1/2, 1/2) = 1/2 \pmod{1} \quad (4.101)$$

Hence we see that $2b(1/2, 1/2) = 1 = 0 \pmod{1}$. So the S matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{4.102}$$

which is clearly singular. It is trivial to see that for any cyclic group of even order there will always be two rows in the S -matrix with 1's in the entries (the first row *always* has 1's in the entries). Hence the S -matrix will be singular in these cases, i.e. $\mathbb{C}[\mathcal{D}]_{(\mathcal{D},q,c)}$ is often not a modular tensor category. These theories *cannot* describe toral Chern-Simons.

Chapter 5

Group Categories

5.1 Introduction

The goal of this chapter is to construct a family of modular tensor categories such that the associated TQFTs are isomorphic to the TQFTs arising from toral (non-spin) Chern-Simons theories.¹ We already saw an easy family of examples in chapter (4) but we argued that these categories do not correspond to toral Chern-Simons.

Here we formulate the underlying braided categories in terms of an explicit set of equations. It turns out that these equations can be cast in the language of abelian group cohomology formulated by Eilenberg and MacLane in the 1940's, hence allowing the use of homology and homotopy theory techniques [EMb]. This identification was studied (in various incarnations) by Frölich and Kerler [FK93], Joyal and Street [JS93], and Quinn [Qui99]. The resulting braided categories are *group categories*. Recently much more work has

¹Belov and Moore produce only part of the data required for an extended $(2 + 1)$ -dim TQFT. We prove an isomorphism of (non-extended) 2-d modular functors in this paper.

been done concerning group categories ² (see for example [ENO05, DGNO07]). The same braiding construction in slightly altered language also appeared in appendix E of [MS89] as well as in [MPR93].

We point out that if the Belov-Moore construction had provided an *extended* 2-d modular functor (see chapter (2)) then we could reverse-engineer the corresponding modular tensor categories. As mentioned more completely in chapter (2) we have the following causal relationships:

$$\begin{array}{ccccc}
 \text{Modular Tensor Category} & \Longrightarrow & \text{Extended } (2+1)\text{-dim TQFT} & \Longrightarrow & (2+1)\text{-dim TQFT} \\
 & & \Downarrow & & \\
 & \swarrow & \text{Extended 2-d Modular Functor} & \searrow & \\
 & & & &
 \end{array}
 \tag{5.1}$$

In this limited sense the modular tensor categories described here *extend and complete* the partial theories introduced in [BM05] using a rather different approach.

From toral Chern-Simons considerations in chapter (3) it was shown that the quantum data is encoded in a finite abelian group \mathcal{D} , a *pure* quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$, and c (an integer mod 24) that encodes a choice of cube root of the Gauss reciprocity formula. Hence we shall use this data to construct a modular tensor category. We remind the reader that we are not considering the more general spin/odd theories considered in [BM05], but rather we are restricted to the even theories because modular tensor categories correspond to *ordinary* TQFTs. We also mention that the third piece of data c will not be necessary. However c can play a role depending on the type of extended structure placed on 3-bordisms [Ati90b, Wal91, FG91]. ³

²We thank Victor Ostrik for useful comments that guided us toward these examples.

³Two examples of an extended structure are a 2-framing and a p_1 -structure. The 2-

5.2 Category $\mathcal{C}_{\mathcal{D}}$ of \mathcal{D} -graded complex vector spaces

Let \mathcal{D} be a finite ⁴ group (not necessarily abelian, but abelian in our case). Following Frölich and Kerler [FK93], Quinn [Qui99], and Joyal and Street [JS93] we consider the following category $\mathcal{C}_{\mathcal{D}}$:

1. $\text{Ob}(\mathcal{C}_{\mathcal{D}})$ consists of finite-dimensional \mathcal{D} -graded complex vector spaces. In other words each object $V \in \text{Ob}(\mathcal{C}_{\mathcal{D}})$ is a finite-dimensional complex vector space that can be decomposed into homogeneously-graded summands $V = \bigoplus_{x \in \mathcal{D}} V_x$.
2. $\text{Mor}(\mathcal{C}_{\mathcal{D}})$ consists of \mathbb{C} -linear maps that respect the group grading (i.e. the only nonzero blocks in a linear map $L : (V = \bigoplus_{x \in \mathcal{D}} V_x) \rightarrow (W = \bigoplus_{y \in \mathcal{D}} W_y)$ are along the diagonal $x = y$).
3. $\mathcal{C}_{\mathcal{D}}$ has a monoidal structure \otimes . If V_x and W_y are homogeneously-graded objects then the product is defined by: $V_x \otimes W_y \equiv (V \otimes W)_{xy}$ (the tensor product on the RHS is the usual one for vector spaces, and the grading obeys the group law). More generally, for non-homogeneously-graded objects if we impose the condition that \otimes distributes over \oplus then the above multiplication formula becomes convolution:

$$(V \otimes W)_z = \bigoplus_{x,y|xy=z} V_x \otimes W_y \tag{5.2}$$

framing is related to the p_1 -structure by a factor of $1/3$, hence this explains why the treatment in [BM05] requires a cube root of the Gauss reciprocity formula whereas we do not. Compare equation (2.1) in [Ati90b] with theorem (2.3) in [FG91].

⁴We limit ourselves to finite groups here, but this is not necessary.

The product of morphisms is defined similarly.

Now let us make explicit some of the properties of $\mathcal{C}_{\mathcal{D}}$:

1. Since the vector space tensor product is *strictly* associative (see chapter (4)) and group multiplication is *strictly* associative we have that $\mathcal{C}_{\mathcal{D}}$ is strictly associative with the identity

$$\bigoplus_{x,y,z|(xy)z=a} (V_x \otimes W_y) \otimes Z_z = \bigoplus_{x,y,z|x(yz)=a} V_x \otimes (W_y \otimes Z_z) \quad (5.3)$$

2. The vector space tensor product always comes equipped with a canonical isomorphism $\text{Perm}_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ defined on vectors by $v \otimes w \mapsto w \otimes v$. This product is *symmetric*, meaning that we have an involution $\text{Perm}_{V,W} \circ \text{Perm}_{V,W} = \text{id}$. More generally, the symmetric group S_n acts on the tensor product of n factors. If we mod out by the action of S_n then we obtain the symmetric tensor product. In this sense the vector space tensor product is commutative.

Now consider the graded picture. For $x, y \in \mathcal{D}$ it is not always true that $xy = yx$, hence $\text{Perm}_{V,W}$ does *not* in general lift to a canonical isomorphism $V_x \otimes W_y \xrightarrow{\sim} W_y \otimes V_x$ (since morphisms by definition must preserve grading). However, if \mathcal{D} is *abelian* then $xy = yx$ and we have an induced canonical isomorphism

$$\overline{\text{Perm}_{V,W}} : V_x \otimes W_y \xrightarrow{\sim} W_y \otimes V_x \quad (5.4)$$

for any $x, y \in \mathcal{D}$.

3. Following the approach outlined in chapter (4) we will shortly abandon

the above associativity and commutativity in favor of a nontrivial family of natural isomorphisms.

4. $\mathcal{C}_{\mathcal{D}}$ is an *abelian* category enriched over \mathbb{C} -vector spaces. This is easily verified as follows: it is clearly preadditive (Ab-category) since the sets $\text{Mor}(\mathcal{C}_{\mathcal{D}})$ are abelian groups (even better they are \mathbb{C} -vector spaces, so we refer to the morphism sets as *Hom* sets from here on).

The \oplus operation makes $\mathcal{C}_{\mathcal{D}}$ an additive category. It is preabelian because any linear map in $\text{Hom}(V, W)$ has a kernel and a cokernel. Finally, it is easy to verify that any injective map $L : V \rightarrow W$ is the kernel of some map (namely the projection $W \rightarrow W/L(V)$); also any surjective map $L : V \rightarrow W$ is the cokernel of the projection map $V \oplus W \rightarrow V$. So $\mathcal{C}_{\mathcal{D}}$ is an abelian category enriched over \mathbb{C} -vector spaces.

5. The monoidal structure on $\mathcal{C}_{\mathcal{D}}$ is compatible with the abelian category structure (i.e. \otimes distributes over \oplus).
6. $\mathcal{C}_{\mathcal{D}}$ is clearly semisimple (every short exact sequence splits). More plainly any object can be decomposed as the direct sum of *simple objects*. The simple objects are 1-dimensional homogeneously-graded vector spaces; we denote them

$$\{\mathbb{C}_x\}_{x \in \mathcal{D}} \quad \text{simple objects.} \quad (5.5)$$

7. There are only finitely-many simple objects since \mathcal{D} is a finite group. In fact it is easy to define left and right duals and interpret $\mathcal{C}_{\mathcal{D}}$ as a fusion category, but we refrain from doing so (we shall only define a right dual below).

8. $\mathcal{C}_{\mathcal{D}}$ can be viewed as the group ring $\text{Vect}_{\mathbb{C}}[\mathcal{D}]$ where the coefficients are finite dimensional complex vector spaces.
9. Alternatively, $\mathcal{C}_{\mathcal{D}}$ can be profitably interpreted as the category of finite dimensional complex vector bundles over \mathcal{D} . The multiplication of two complex vector bundles is defined to be the pushforward along multiplication on the base space \mathcal{D} (i.e. convolution).

The category $\mathcal{C}_{\mathcal{D}}$ is the canonical example of a *group category*:

Definition 5.6. A **group category**⁵ is a category with the following additional structure:

1. Additive \oplus
2. Monoidal \otimes
3. \otimes distributes over \oplus .
4. Each Hom space is an complex vector space.⁶
5. An object V is called *simple* if $\text{Hom}(V, V) \cong \mathbb{C}$. Group categories are required to be semisimple (any object can be decomposed as a *finite* sum of simple objects - however there need *not* be finitely-many simple objects).

⁵We follow Quinn's definition [Qui99] which has an additive structure that does not appear in the "categorical groups" discussed in Joyal and Street [JS93] (the *only* objects in [JS93] are simple). However by adding a formal \oplus it is trivial to recover Quinn's definition.

⁶Quinn points out that it is often necessary to work with R -modules where R is a commutative ground ring. We do not need that greater generality here.

6. For each simple object V we require a *right dual* object V^* and a distinguished *isomorphism* $d_V : V^* \otimes V \rightarrow \mathbb{1}$ where $\mathbb{1}$ is the unit object for the monoidal structure.⁷
7. If V and W are distinct simple objects then we require $\text{Hom}(V, W) \cong 0$.

We note that the existence of a distinguished *isomorphism* d_V for each simple object is a strong condition. We say that the simple objects are *invertible*. It is straightforward to check that the definition implies that if V and W are simple then $V \otimes W$ is simple. In other words the simple objects form a group - the *underlying* group of the group category.

From here on we limit ourselves to the situation where \mathcal{D} is a finite abelian group.

5.3 Twisted version $\mathcal{C}_{(\mathcal{D}, q)}$: nontrivial associativity and braiding

In the last section we introduced the category $\mathcal{C}_{\mathcal{D}}$. We mentioned that if the underlying group \mathcal{D} is abelian then $\mathcal{C}_{\mathcal{D}}$ is commutative in the sense that the tensor product of n objects admits an action of the symmetric group S_n . Furthermore the monoidal structure is strict. *Since we are dealing with finite abelian groups from now on we switch from multiplicative xy to additive $x + y$ notation.*

In light of chapter (4) we aim to twist the structure described in the last section to produce a non-strict modular tensor category. Since the quantum

⁷This is the “ d_V ” map that is part of the definition of duality. However here it is an isomorphism rather than just a morphism. We did not mention this for the example $\mathcal{C}_{\mathcal{D}}$, but we shall mention it below.

data for toral Chern-Simons is encoded in the trio (\mathcal{D}, q, c) we expect to use this data to twist the structure appropriately (however we shall not require c in this chapter). In light of this we denote the resulting twisted category $\mathcal{C}_{(\mathcal{D}, q)}$. Interestingly, a fixed set of data (\mathcal{D}, q) actually produces a *family* of modular tensor categories. We shall discuss how MTCs in a given family are related to each other.

Since $\mathcal{C}_{(\mathcal{D}, q)}$ is an additive category it suffices to confine our study to the simple objects

$$\{\mathbb{C}_x\}_{x \in \mathcal{D}} \quad (5.7)$$

(we can extend to arbitrary objects by additivity). The fusion rules are trivial because of the strong structure imposed by a group category:

$$\mathbb{C}_x \otimes \mathbb{C}_y \cong \mathbb{C}_{x+y} \quad (5.8)$$

Let us first consider relaxing the associativity *identity* in equation (5.3) and allow instead a family of natural isomorphisms

$$\{a_{x,y,z} : (\mathbb{C}_x \otimes \mathbb{C}_y) \otimes \mathbb{C}_z \xrightarrow{\sim} \mathbb{C}_x \otimes (\mathbb{C}_y \otimes \mathbb{C}_z)\}_{x,y,z \in \mathcal{D}} \quad (5.9)$$

Since the tensor product of simple objects is simple, for fixed $x, y, z \in \mathcal{D}$ this is just an endomorphism

$$a_{x,y,z} : \mathbb{C}_{x+y+z} \xrightarrow{\sim} \mathbb{C}_{x+y+z} \quad (5.10)$$

In other words for each $x, y, z \in \mathcal{D}$ it suffices to specify a complex number $a_{x,y,z}$ (we have reused notation) such for $v \in \mathbb{C}_{x+y+z}$ we have $v \mapsto a_{x,y,z}v$.

It is clear that the unit object is just $\mathbb{1} \equiv \mathbb{C}_0$. In order to find the coefficients $a_{x,y,z}$ we impose the pentagon identity (equation (4.15)) and the triangle identity (equation (4.16)). Since all isomorphisms involved are merely multiplication by complex numbers we need not be concerned with ordering. Explicitly, for $v_x \in \mathbb{C}_x$, $v_y \in \mathbb{C}_y$, $v_z \in \mathbb{C}_z$, and $v_w \in \mathbb{C}_w$ we follow the upper part of the pentagon diagram:

$$\begin{aligned} ((v_x \otimes v_y) \otimes v_z) \otimes v_w &\mapsto a_{x+y,z,w}(v_x \otimes v_y) \otimes (v_z \otimes v_w) \mapsto \\ &a_{x,y,z+w}a_{x+y,z,w}v_x \otimes (v_y \otimes (v_z \otimes v_w)) \end{aligned} \quad (5.11)$$

Following the lower part of the pentagon diagram gives us

$$\begin{aligned} ((v_x \otimes v_y) \otimes v_z) \otimes v_w &\mapsto a_{x,y,z}(v_x \otimes (v_y \otimes v_z)) \otimes v_w \mapsto \\ a_{x,y+z,w}a_{x,y,z}v_x \otimes ((v_y \otimes v_z) \otimes v_w) &\mapsto a_{y,z,w}a_{x,y+z,w}a_{x,y,z}v_x \otimes (v_y \otimes (v_z \otimes v_w)) \end{aligned} \quad (5.12)$$

Comparing these we see that

$$a_{x,y,z+w}a_{x+y,z,w} = a_{y,z,w}a_{x,y+z,w}a_{x,y,z} \quad (5.13)$$

If we restrict ourselves to solutions living in the unit circle then we can write

$$a_{x,y,z} := \exp(2\pi ih(x, y, z)) \quad (5.14)$$

for a phase function $h : \mathcal{D}^3 \rightarrow \mathbb{Q}/\mathbb{Z}$. Equation (5.13) becomes

$$h(x, y, z + w) + h(x + y, z, w) \equiv h(y, z, w) + h(x, y + z, w) + h(x, y, z) \pmod{1} \quad (5.15)$$

Now let us consider the triangle diagram in equation (4.16). If we set the right and left identity maps in equations (4.13) and (4.14) to be just multiplication by 1, then the triangle diagram implies

$$a_{x,0,y} = 1 \quad (5.16)$$

In terms of h this is just (mod 1)

$$h(x, 0, y) = 0 \quad (5.17)$$

It is easy to exploit equation (5.15) to then prove that (mod 1)

$$h(x, 0, y) = h(0, x, y) = h(x, y, 0) = 0 \quad (5.18)$$

Now we wish to consider the hexagon relations depicted in equations (4.24) and (4.25). For simple objects \mathbb{C}_x and \mathbb{C}_y we postulate a braiding isomorphism meant to replace the involution Perm: ⁸

$$c_{x,y} : \mathbb{C}_x \otimes \mathbb{C}_y \xrightarrow{\sim} \mathbb{C}_y \otimes \mathbb{C}_x \quad (5.19)$$

Again, because of the trivial fusion rules $\mathbb{C}_x \otimes \mathbb{C}_y \cong \mathbb{C}_{x+y}$ this is effectively an

⁸this can be extended to arbitrary objects by linearity

isomorphism

$$c_{x,y} : \mathbb{C}_{x+y} \xrightarrow{\sim} \mathbb{C}_{x+y} \quad (5.20)$$

and hence is determined by a 1×1 complex matrix $[c_{x,y}]$. Continuing with our previous restriction to coefficients living in the unit circle

$$c_{x,y} := \exp(2\pi i s(x,y)) \quad s : \mathcal{D}^2 \rightarrow \mathbb{Q}/\mathbb{Z} \quad (5.21)$$

we see that the hexagon relations imply (mod 1)

$$s(x, y+z) = -h(x, y, z) + s(x, y) + h(y, x, z) + s(x, z) - h(y, z, x) \quad (5.22)$$

$$s(x+y, z) = h(x, y, z) + s(y, z) - h(x, z, y) + s(x, z) + h(z, x, y)$$

As was the case for the function h , it is easy to calculate using these identities that

$$s(0, y) = s(x, 0) = 0 \quad (5.23)$$

Summarizing, we can twist the category $\mathcal{C}_{\mathcal{D}}$ into a braided group category $\mathcal{C}_{(\mathcal{D},q)}$ by relaxing the associativity and commutativity identities. We still expect that any reasonable theory should obey the pentagon, triangle, and hexagon relations as described in chapter (4). Since the fusion rules are rather simple these relations can be cast into the form of equations (5.15), (5.18), and (5.22) which are valued in \mathbb{Q}/\mathbb{Z} .

An interesting observation is that typically there are multiple solutions to these equations (that turn out to be braided monoidal equivalent). Since there are multiple solutions we denote the group category associated to a

solution (h, s) by the notation

$$\mathcal{C}_{(\mathcal{D}, q)}(h, s) \tag{5.24}$$

This provides a richer structure than one might naively expect. In the next section following Frölich and Kerler [FK93], Quinn [Qui99], and Joyal and Street [JS93] we identify these equations as cocycles in group cohomology of abelian groups and provide explicit solutions.

5.4 Connection with group cohomology

In this section we provide a brief outline of *abelian* group cohomology as introduced by Eilenberg and MacLane (see [EMa] for a brief introduction and [EMb] for a more detailed account).⁹

Before we begin fix an underlying group Π (in our case we will be considering the finite abelian group \mathcal{D}). Fix an integer m and an abelian coefficient group H (in our case $H = \mathbb{Q}/\mathbb{Z}$).

Consider a path-connected topological space X such that $\pi_m(X) \cong \Pi$ and all other homotopy groups are trivial (clearly if $m > 1$ then Π must be abelian). We wish to study the homology and cohomology groups of this space. One of the fundamental results of Eilenberg and MacLane is that if Y is a different topological space with the same homotopy groups then the

⁹Warning: the conventions used by Quinn [Qui99] do not follow those of Eilenberg and MacLane. In particular the dimensions of the cells in the relevant complex are defined to be 1 dimension higher in Quinn's paper. Hence there he studies H^4 whereas the *same* cohomology classes are in H^3 in the other references.

homology (cohomology) groups are also the same:

$$H(X; H) \cong H(Y; H) \tag{5.25}$$

This implies that it suffices to study the homology and cohomology groups of the *standard* Eilenberg-MacLane space $K(\Pi, m)$ (a cell complex explicitly constructed below such that $\pi_m(X) \cong \Pi$ and all other homotopy groups are trivial).

On the other hand the main point of [EMa] and [EMb] is that if Π is abelian (it is in our case) then the cell complex $K(\Pi, m)$ can be replaced by a cell complex $A(\Pi)$ such that the cohomology groups $H^k(A(\Pi); H)$ are much simpler to compute. By “replace” we mean that the following isomorphism holds ([EMa] article II, Theorem 6):

$$H^{m-1+k}(K(\Pi, m); H) \cong H^k(A(\Pi); H) \quad k = 1, \dots, m \tag{5.26}$$

We note that m does not appear on the RHS (and $A(\Pi)$ is independent of m). However, the isomorphism only holds for $k \leq m$.

We will eventually be interested in the case when $m = 2$ and $k = 3$, which clearly does not satisfy the requirement $k \leq m$. However, a more general statement can be made as follows. The space $A(\Pi)$ is constructed iteratively using the *bar* construction B . That is we have a sequence of embedded spaces

$$A^0(\Pi) \subset A^1(\Pi) \subset A^2(\Pi) \subset \dots \subset A^\infty(\Pi) \tag{5.27}$$

where we start with $A^0(\Pi) = K(\Pi, 1)$ and apply the iterated bar construction (see below) $A^1(\Pi) = B(A^0(\Pi))$, $A^2(\Pi) = B(A^1(\Pi)) = B(B(A^0(\Pi)))$, etc. We

define $A(\Pi) = A^\infty(\Pi)$.

Now for *arbitrary* k the following isomorphism holds:

$$H^{m-1+k}(K(\Pi, m); H) \cong H^k(A^{m-1}(\Pi); H) \quad (5.28)$$

which is compatible with the previous isomorphism in the sense that

$$H^k(A^{m-1}(\Pi); H) \cong H^k(A(\Pi); H) \quad k = 1, \dots, m \quad (5.29)$$

For $m = 2$ and $k = 3$ (our case of interest) this is just

$$H^4(K(\Pi, 2); H) \cong H^3(A^1(\Pi); H) \quad (5.30)$$

Now let us discuss the iterated bar construction which will demonstrate why we are interested in $H^3(A^1(\mathcal{D}); \mathbb{Q}/\mathbb{Z})$.

Iterated bar construction

Since the iterated bar construction bootstraps using $K(\Pi, 1)$ we construct this cell complex first. Provide a q -dimensional cell labelled $[x_1, \dots, x_q]$ for each q -tuple of elements $x_1, \dots, x_q \in \Pi$. This cell attaches to the $(q - 1)$ -skeleton using the boundary operator

$$\begin{aligned} \partial[x_1, \dots, x_q] = [x_2, \dots, x_q] + \sum_{i=1}^{q-1} (-1)^i [x_1, \dots, x_i x_{i+1}, \dots, x_q] \\ + (-1)^q [x_1, \dots, x_{q-1}] \quad (5.31) \end{aligned}$$

(for 1-cells the boundary formula is defined as $\partial[x] = 0$ since each endpoint will attach to the unique 0-cell []).

From now on we refer to $K(\Pi, 1)$ as $A^0(\Pi)$. We define a product $*_0$ on the cells of $A^0(\Pi)$ via *shuffling* (extend this to chains by bilinearity):

$$[x_1, \dots, x_q] *_0 [y_1, \dots, y_r] = \sum (-1)^\epsilon [z_1, \dots, z_{q+r}] \quad (5.32)$$

Here we are summing over all of the shuffles of the list $\{x_1, \dots, x_q, y_1, \dots, y_r\}$ where the x_i 's must stay in order relative to each other, and likewise for the y_i 's (i.e. x 's can only swap with y 's). The sign $(-1)^\epsilon$ is 1 if the total number of transpositions is even, and -1 if the total number of transpositions is odd.

For abelian Π the operation $*_0$ defines a *product of excess 0*. In general a *product of excess k* on a chain complex is a bilinear function $*_k$ on chains a and b such that if $d(a)$ denotes the *cell* dimension of a then

$$d(a *_k b) = d(a) + d(b) + k \quad (5.33)$$

If we define $d_k(a) = d(a) + k$ then this can be written more suggestively as

$$d_k(a *_k b) = d_k(a) + d_k(b) \quad (5.34)$$

In addition we require a product of excess k to be associative, graded commu-

tative, and behave as usual with respect to the boundary operator:

$$a *_k (b *_k c) = (a *_k b) *_k c \quad (5.35)$$

$$b *_k a = (-1)^\epsilon a *_k b \quad \epsilon = d_k(a)d_k(b) \quad (5.36)$$

$$\partial(a *_k b) = (\partial a) *_k b + (-1)^{d_k(a)} a *_k (\partial b) \quad (5.37)$$

We iteratively define the complexes $A^*(\Pi)$ as follows: from the complex $A^{k-1}(\Pi)$ with the product $*_{k-1}$ of excess $k-1$ we can produce a complex $A^k(\Pi)$ which contains $A^{k-1}(\Pi)$ and in addition contains new cells written

$$[a_1|_k \dots |_k a_p] \quad a_i \text{ are cells of } A^k(\Pi) \quad (5.38)$$

These cells are defined to have cell dimension

$$d([a_1|_k \dots |_k a_p]) = d(a_1) + \dots + d(a_p) + (p-1)k \quad (5.39)$$

In practice we write $|_1 = |$, $|_2 = ||$, etc.

The boundary operator is defined as

$$\begin{aligned} \partial[a_1|_k \dots |_k a_p] = & \sum_{i=1}^p (-1)^{\epsilon_{i-1}} [a_1|_k \dots |_k a_{i-1}|_k \partial a_i|_k a_{i+1}|_k \dots a_p] + \\ & \sum_{i=1}^{p-1} (-1)^{\epsilon_i} [a_1|_k \dots |_k a_{i-1}|_k a_i *_k a_{i+1}|_k a_{i+2}|_k \dots a_p] \end{aligned} \quad (5.40)$$

where $\epsilon_i = d_k(a_1) + \dots + d_k(a_i)$.

We can also define a product of excess k on $A^k(\Pi)$ using a similar shuffle

construction

$$[a_1|_k \dots |_k a_p] *_k [b_1|_k \dots |_k b_r] = \sum (-1)^\epsilon [z_1|_k \dots |_k z_{p+r}] \quad (5.41)$$

where ϵ can be determined via the rule: a transposition of a and b multiplies by a factor $(-1)^{d_k(a)d_k(b)}$.¹⁰

$$H^3(A^1(\Pi); H)$$

Using these constructions it is simple to write down the cells in $A^1(\Pi)$ (we will only bother up through cell dimension 4):

- dimension 0: $[\]$
- dimension 1: $[x]$ where $x \in \Pi$
- dimension 2: $[x, y]$ where $x, y \in \Pi$
- dimension 3: $[x, y, z]$ and $[x|y]$ where $x, y, z \in \Pi$
- dimension 4: $[x, y, z, w]$, $[x, y|z]$, and $[x|y, z]$ where $x, y, z, w \in \Pi$

The boundaries are easily computed:

- dimension 0: $\partial[\] = 0$
- dimension 1: $\partial[x] = 0$
- dimension 2: $\partial[x, y] = [y] - [x + y] + [x]$

¹⁰again the a 's must stay in order relative to each other, and likewise for the b 's.

- dimension 3:

$$\begin{aligned}\partial[x, y, z] &= [y, z] - [x + y, z] + [x, y + z] - [x, y] \\ \partial[x|y] &= [x, y] - [y, x]\end{aligned}\tag{5.42}$$

- dimension 4:

$$\begin{aligned}\partial[x, y, z, w] &= [y, z, w] - [x + y, z, w] \\ &\quad + [x, y + z, w] - [x, y, z + w] + [x, y, z] \\ \partial[x, y|z] &= [\partial[x, y]|z] - [[x, y] *_{\mathbf{0}} z] \\ &= [y|z] - [x + y|z] + [x|z] - [x, y, z] + [x, z, y] - [z, x, y] \\ \partial[x|y, z] &= [x|\partial[y, z]] - [x *_{\mathbf{0}} [y, z]] \\ &= [x|z] - [x|y + z] + [x|y] - [x, y, z] + [y, x, z] - [y, z, x]\end{aligned}\tag{5.43}$$

This provides a characterization of homology. Now let us compute cohomology. We are only interested in $H^3(A^1(\Pi); H)$. Consider a 3-cochain (a homomorphism)

$$f : \text{3-chains} \rightarrow H\tag{5.44}$$

When restricted to 3-cells of the form $[x, y, z]$ we use the notation

$$h(x, y, z) := f([x, y, z])\tag{5.45}$$

When restricted to 3-cells of the form $[x|y]$ we use the notation

$$s(x, y) := f([x|y])\tag{5.46}$$

To compute the cocycle condition $\delta f = 0$ it is easy to write out the condition

$$(\delta f)([4\text{-chain}]) := f(\partial[4\text{-chain}]) = 0 \quad (5.47)$$

and then use the boundary formulas in equation (5.43). If we consider the case where $\Pi = \mathcal{D}$ and the coefficient group $H = \mathbb{Q}/\mathbb{Z}$ then this obviously reproduces equations (5.15) and (5.22).

The only condition left to encode is the triangle identity (and its consequences) in equation (5.18). For convenience we copy the conditions again:

$$h(x, 0, z) = h(x, y, 0) = h(0, y, z) = s(x, 0) = s(0, y) = 0 \quad (5.48)$$

This is straightforward to achieve with cohomology of *normalized chains*. Let $A_N^1(\Pi)$ be the subcomplex of $A^1(\Pi)$ consisting of cells $[x_1, \dots, x_q]$ with at least one $x_i = 0$ (any of the commas may be replaced with bars $|$ as well). Then all of the identities are satisfied by the cohomology of *normalized 3-cochains*¹¹

$$H^3(A^1(\mathcal{D})/A_N^1(\mathcal{D}); \mathbb{Q}/\mathbb{Z}) \quad (5.49)$$

We will refrain from over-decorating the notation with cohomology of normalized chains since it does not affect the outcome.

Explicit cocycles

The groups $H^3(A^1(\Pi); H)$ were computed in the original Eilenberg-MacLane articles (see [EMb] article II pg 92 and pg 130). For Π cyclic an explicit

¹¹A similar subcomplex $K_N(\Pi, m)$ of $K(\Pi, m)$ can be defined and cohomology can be studied there as well.

computation is performed in [JS93] and the full computation for general finite abelian groups can be found in [Qui99]. For the reader who wishes to compare the different references we emphasize again the following isomorphism:

$$H^4(K(\Pi, 2); H) \cong H^3(A^1(\Pi); H) \quad (5.50)$$

Let $q_1 : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ and $q_2 : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ be two pure quadratic forms. Then it is easy to verify that $q_1 + q_2$ is also a pure quadratic form. It is also trivial to verify that for a pure quadratic form q its inverse $-q$ is also a pure quadratic form. Finally the constant function $q = 0$ is also a pure quadratic form. Hence the set of pure quadratic forms

$$\{q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}\} \quad (5.51)$$

forms a group which we denote by $\text{Quad}(\mathcal{D}, \mathbb{Q}/\mathbb{Z})$. It is shown in [EMb] pg 130 that there is a canonical isomorphism

$$H^3(A^1(\mathcal{D}); \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \text{Quad}(\mathcal{D}, \mathbb{Q}/\mathbb{Z}) \quad (5.52)$$

determined by defining $q(x) := s(x, x)$.

What we are missing is a recipe that produces an explicit representative cocycle (h, s) from a finite abelian group \mathcal{D} equipped with a quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$. Following Quinn [Qui99] we have the following (family of) explicit solutions:

1. Pick a set of generators 1_i for \mathcal{D} (\mathcal{D} is a finite abelian group, hence can be decomposed into cyclic factors of order n_i)

2. Pick an *ordering* of the generators $1_1 < 1_2 < \dots$
3. Write any arbitrary element $x \in \mathcal{D}$ as $x = a_1 1_1 + a_2 1_2 + \dots$ *such that*
 $0 \leq a_i < n_i$ *for every* i

We emphasize that this construction is *not* well defined on the group \mathcal{D} , but is well defined on the group \mathcal{D} *equipped with ordered generators*. For further emphasis we repeat that the coefficients a_i **must always be written as integers** $0 \leq a_i < n_i$ (i.e. we do *not* write $-x = -a_1 1_1 - a_2 1_2 - \dots$, but rather $-x = (n_1 - a_1) 1_1 + (n_2 - a_2) 1_2 + \dots$).

Since \mathcal{D} is equipped with a quadratic form q we denote $q_i := q(1_i) \in \mathbb{Q}/\mathbb{Z}$. Also a pure quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ determines a bilinear form $b : \mathcal{D} \otimes \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ defined by $b(x, y) := q(x + y) - q(x) - q(y)$. We denote $b_{ij} := b(1_i, 1_j)$.

Then if $x = \sum_i a_i 1_i$, $y = \sum_i b_i 1_i$, and $z = \sum_i c_i 1_i$ then the associativity is defined by

$$h(x, y, z) = \sum_i \begin{cases} 0 & \text{if } b_i + c_i < n_i \\ n_i a_i q_i & \text{if } b_i + c_i \geq n_i \end{cases} \quad (5.53)$$

and the braiding is given by

$$s(x, y) = \sum_{i < j} a_i b_j b_{ij} + \sum_i a_i b_i q_i \quad (5.54)$$

Some quick calculations confirm that this solution satisfies equations (5.15), (5.18), and (5.22).

Coboundaries and braided monoidal equivalence

We mentioned in the last section that a cohomology class $[h, s]$ is determined by a quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$, and to find an explicit representative (h, s) we are forced to pick an ordered set of generators.

The isomorphism (proved in [EMb])

$$H^3(A^1(\mathcal{D}); \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \text{Quad}(\mathcal{D}, \mathbb{Q}/\mathbb{Z}) \quad (5.55)$$

means that if we have two representatives (h, s) and (h', s') that are determined by different choices of ordered generators then their difference $(h, s) - (h', s')$ must be a coboundary. This is easy to show directly: if we consider the homology boundary maps in equation (5.42) then passing to cohomology the expression $(h, s) - (h', s')$ should be the coboundary of *some* function $k : \mathcal{D}^2 \rightarrow \mathbb{Q}/\mathbb{Z}$, i.e.

$$\begin{aligned} (h - h')(x, y, z) &= k(y, z) - k(x + y, z) + k(x, y + z) - k(x, y) \\ (s - s')(x, y) &= k(x, y) - k(y, x) \end{aligned} \quad (5.56)$$

A tedious calculation shows that for (h, s) , (h', s') determined by different choices of ordered generators there is such a function k .

Now we must answer how two group categories $\mathcal{C}_{(\mathcal{D}, q)}(h, s)$ and $\mathcal{C}_{(\mathcal{D}, q)}(h', s')$ constructed from cohomologous (h, s) and (h', s') are related. It turns out that the resulting group categories are *braided monoidal equivalent*. This was proven by Joyal and Street [JS93] (the proof is written in slightly greater detail below).

In order to define a braided monoidal equivalence we start with some

preliminaries.

Definition 5.57. Let $\mathcal{V}, \mathcal{V}'$ be two monoidal categories. A **monoidal functor** is a triple (F, ϕ_2, ϕ_0) given by [JS93]

1. A functor $F : \mathcal{V} \rightarrow \mathcal{V}'$.
2. A family of natural isomorphisms (one for each pair of objects $A, B \in \mathcal{V}$):

$$\phi_{2,A,B} : FA \otimes FB \xrightarrow{\sim} F(A \otimes B) \quad (5.58)$$

3. An isomorphism

$$\phi_0 : \mathbb{1}' \xrightarrow{\sim} F\mathbb{1} \quad (5.59)$$

In addition we require that the following diagrams commute:

$$\begin{array}{ccc}
 & FA \otimes (FB \otimes FC) & \\
 a_{A,B,C} \nearrow & & \searrow \text{id}_A \otimes \phi_{2,B,C} \\
 (FA \otimes FB) \otimes FC & & FA \otimes F(B \otimes C) \\
 \downarrow \phi_{2,A,B} \otimes \text{id}_C & & \downarrow \phi_{2,A,B \otimes C} \\
 F(A \otimes B) \otimes FC & & F(A \otimes (B \otimes C)) \\
 \searrow \phi_{2,A \otimes B,C} & & \nearrow F(a_{A,B,C}) \\
 & F((A \otimes B) \otimes C) &
 \end{array} \quad (5.60)$$

$$\begin{array}{ccc}
 FA \otimes \mathbb{1}' \xrightarrow{r_{FA}} FA & & \mathbb{1}' \otimes FA \xrightarrow{l_{FA}} FA \\
 \downarrow \text{id}_{FA} \otimes \phi_0 & F(r_A) \uparrow & \downarrow \phi_0 \otimes \text{id}_{FA} & F(l_A) \uparrow \\
 FA \otimes F\mathbb{1} \xrightarrow{\phi_{2,A,\mathbb{1}}} F(A \otimes \mathbb{1}) & & F\mathbb{1} \otimes FA \xrightarrow{\phi_{2,\mathbb{1},A}} F(\mathbb{1} \otimes A)
 \end{array} \quad (5.61)$$

We can have natural transformations (and natural isomorphisms) between ordinary functors; we want to extend to a notion of *monoidal natural*

transformation between two *monoidal* functors.

Definition 5.62. Let $F : \mathcal{V} \rightarrow \mathcal{V}'$ and $G : \mathcal{V} \rightarrow \mathcal{V}'$ be monoidal functors. A **monoidal natural transformation** is an ordinary natural transformation $\theta : F \rightarrow G$ that in addition is required to satisfy the following commutative diagrams:

$$\begin{array}{ccc}
 FA \otimes FB & \xrightarrow{\phi_{2,A,B}^F} & F(A \otimes B) \\
 \downarrow \theta_A \otimes \theta_B & & \downarrow \theta_{A \otimes B} \\
 GA \otimes GB & \xrightarrow{\phi_{2,A,B}^G} & G(A \otimes B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & F\mathbb{1} \\
 & \nearrow \phi_0^F & \downarrow \theta_{\mathbb{1}} \\
 \mathbb{1}' & & G\mathbb{1} \\
 & \searrow \phi_0^G &
 \end{array}
 \tag{5.63}$$

This defines a **monoidal natural isomorphism** if all of the arrows θ_A are isomorphisms. We denote a monoidal natural isomorphism by the symbol \cong .

Now define a notion of equivalence between two monoidal categories:

Definition 5.64. Let $(F, \phi_2^F, \phi_0^F) : \mathcal{V} \rightarrow \mathcal{V}'$ and $(F', \phi_2^{F'}, \phi_0^{F'}) : \mathcal{V}' \rightarrow \mathcal{V}$ be monoidal functors. Then these are said to be a **monoidal equivalence** if

$$F' \circ F \cong I_{\mathcal{V}} \qquad F \circ F' \cong I_{\mathcal{V}'} \tag{5.65}$$

where $I_{\mathcal{V}}, I_{\mathcal{V}'}$ are the identity monoidal functors.

Now we are ready to consider *braided* monoidal categories.

Definition 5.66. Let \mathcal{V} and \mathcal{V}' be braided monoidal categories with braidings c and c' , respectively (in the sense of chapter (4)). A **braided monoidal functor** $F : \mathcal{V} \rightarrow \mathcal{V}'$ is a monoidal functor that in addition must make the

following compatibility diagram commute:

$$\begin{array}{ccc}
FA \otimes FB & \xrightarrow{\phi_{2,A,B}} & F(A \otimes B) \\
\downarrow c'_{FA,FB} & & \downarrow F(c_{A,B}) \\
FB \otimes FA & \xrightarrow{\phi_{2,B,A}} & F(B \otimes A)
\end{array} \tag{5.67}$$

Definition 5.68. A **braided natural transformation** between two braided monoidal functors $F : \mathcal{V} \rightarrow \mathcal{V}'$ and $G : \mathcal{V} \rightarrow \mathcal{V}'$ is a monoidal natural transformation $\theta : F \rightarrow G$ that satisfies the following compatibility commutative diagram:

$$\begin{array}{ccc}
FA \otimes FB & \xrightarrow{c'_{FA,FB}} & FB \otimes FA \\
\downarrow \theta_A \otimes \theta_B & & \downarrow \theta_B \otimes \theta_A \\
GA \otimes GB & \xrightarrow{c'_{GA,GB}} & GB \otimes GA
\end{array} \tag{5.69}$$

Obviously this defines a **braided monoidal natural isomorphism** if all of the arrows θ_A are isomorphisms. We reuse notation and denote this \cong .

Definition 5.70. Let $(F, \phi_2^F, \phi_0^F) : \mathcal{V} \rightarrow \mathcal{V}'$ and $(F', \phi_2^{F'}, \phi_0^{F'}) : \mathcal{V}' \rightarrow \mathcal{V}$ be braided monoidal functors. Then these are said to be a **braided monoidal equivalence** if

$$F' \circ F \cong I_{\mathcal{V}} \quad F \circ F' \cong I_{\mathcal{V}'}$$
(5.71)

where $I_{\mathcal{V}}, I_{\mathcal{V}'}$ are the identity braided monoidal functors.

Two braided monoidal categories that are braided monoidal equivalent are (in the above sense) the same. This is the appropriate way to interpret the following theorem which answers how to relate group categories constructed by choosing different ordered lists of generators.

Theorem 5.72 (Joyal and Street). *The group categories $\mathcal{C}_{(\mathcal{D},q)}(h, s)$ and $\mathcal{C}_{(\mathcal{D},q)}(h', s')$*

are braided monoidal equivalent iff (h, s) and (h', s') are cohomologous 3-cocycles in $H^3(A^1(\mathcal{D}); \mathbb{Q}/\mathbb{Z})$.

Proof. (\Leftarrow) Suppose that (h, s) and (h', s') are cohomologous, i.e. let $k : \mathcal{D}^2 \rightarrow \mathbb{Q}/\mathbb{Z}$ be as in equation (5.56). Since both categories share the same underlying ordinary category we consider the identity functor

$$I : \mathcal{C}_{(\mathcal{D}, q)}(h, s) \rightarrow \mathcal{C}_{(\mathcal{D}, q)}(h', s') \quad (5.73)$$

This functor is not yet a monoidal functor because the associativity structures h and h' are different. We need to construct ϕ_2 and ϕ_0 .

It is enough to consider the simple objects and extend by linearity. Let $x, y \in \mathcal{D}$. Then the map ¹²

$$\phi_{2,x,y} : \mathbb{C}_x \otimes \mathbb{C}_y \xrightarrow{\sim} \mathbb{C}_x \otimes \mathbb{C}_y \quad \text{multiplication by } \exp(2\pi i k(x, y)) \quad (5.74)$$

and the map

$$\phi_0 : \mathbb{1} \xrightarrow{\sim} \mathbb{1} \quad \text{multiplication by } 1 \quad (5.75)$$

define a monoidal functor (I, ϕ_2, ϕ_0) since it is straightforward to verify that the diagram in equation (5.60) is encoded in the first line of equation (5.56) (and the other diagrams are trivial).

In fact (I, ϕ_2, ϕ_0) also defines a braided monoidal functor because the diagram in equation (5.67) is seen to be encoded in the second line of equation (5.56).

Finally, (I, ϕ_2, ϕ_0) and its obvious inverse $(I, \phi_2^{-1}, \phi_0^{-1})$ are verified (trivially) to form a braided monoidal equivalence.

¹²the source and target are the same since we are using the identity functor

(\Rightarrow) Straightforward using essentially the reverse argument to produce k (left to the reader since we shall not use this result). \square

5.5 Modular tensor category

The categories $\mathcal{C}_{(\mathcal{D},q)}(h, s)$ are braided (non-strict) monoidal categories. In addition we have seen that they are finitely-semisimple abelian categories.¹³ In this section we slightly extend the categories $\mathcal{C}_{(\mathcal{D},q)}(h, s)$ to produce modular tensor categories (we use the same notation since no additional data is required). We do not know if this appears explicitly elsewhere in the literature.

Ribbon structure

First, it is necessary to form a ribbon structure on $\mathcal{C}_{(\mathcal{D},q)}(h, s)$. We start with the twist.

Twisting

Note that the quadratic form satisfies $q(x) := s(x, x)$. For a simple object \mathbb{C}_x we define the twist to be

$$\begin{aligned} \theta_x : \mathbb{C}_x &\rightarrow \mathbb{C}_x & (5.76) \\ v &\mapsto \exp(2\pi i q(x))v \end{aligned}$$

This can be extended to arbitrary objects by linearity. We need to check the balancing identity in equation (4.29).

¹³They are also enriched over \mathbb{C} -vector spaces, i.e. the Hom sets are \mathbb{C} -vector spaces. Furthermore the monoidal structure and the abelian structure are compatible in the sense that \otimes distributes over \oplus .

Proposition 5.77. *The braided monoidal category $\mathcal{C}_{(\mathcal{D},q)}(h, s)$ with twisting defined on the simple objects \mathbb{C}_x by*

$$\begin{aligned}\theta_x : \mathbb{C}_x &\rightarrow \mathbb{C}_x \\ v &\mapsto \exp(2\pi i q(x))v\end{aligned}\tag{5.78}$$

is balanced.

Proof. We check this only on the simple objects. Let \mathbb{C}_x and \mathbb{C}_y be two simple objects. Since $\mathbb{C}_x \otimes \mathbb{C}_y \cong \mathbb{C}_{x+y}$ what we are trying to verify is the equation

$$\theta_{x+y}\theta_x^{-1}\theta_y^{-1} = c_{y,x} \circ c_{x,y}\tag{5.79}$$

The LHS is easy to write out as

$$\exp[2\pi i(q(x+y) - q(x) - q(y))]\tag{5.80}$$

However, because q is a quadratic form we have $q(x+y) - q(x) - q(y) = b(x, y)$ where $b : \mathcal{D} \otimes \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ is the induced bilinear form.

In terms of the generators 1_i for \mathcal{D} we can write

$$\begin{aligned}x &= \sum_i a_i 1_i \\ y &= \sum_j b_j 1_j\end{aligned}\tag{5.81}$$

in which case $b(x, y)$ becomes

$$\sum_{i,j} a_i b_j b(1_i, 1_j)\tag{5.82}$$

which in the notation preceding equation (5.54) is

$$\sum_{i,j} a_i b_j b_{ij} \tag{5.83}$$

which is

$$= 2 \sum_{i < j} a_i b_j b_{ij} + \sum_i a_i b_i b_{ii} \tag{5.84}$$

(we have used the symmetry of $b(\cdot, \cdot)$). However the general relation $q(x + y) - q(x) - q(y) = b(x, y)$ specializes when $x = y$ to $q(2x) - 2q(x) = b(x, x)$, and since q is a pure quadratic form we see that this is just $4q(x) - 2q(x) = 2q(x) = b(x, x)$. In particular $b_{ii} = 2q_i$. In light of this the expression above becomes

$$2 \sum_{i < j} a_i b_j b_{ij} + 2 \sum_i a_i b_i q_i \tag{5.85}$$

which is clearly equal (after taking the exponent) to the RHS $c_{y,x} \circ c_{x,y}$ using the braiding in equation (5.54). \square

Rigidity

Now let us address rigidity. Again, by linearity it suffices to restrict our attention to the simple objects \mathbb{C}_x . Given a simple object \mathbb{C}_x the right dual is

$$(\mathbb{C}_x)^* := \mathbb{C}_{-x} \tag{5.86}$$

Pick a basis v_x for each $\{\mathbb{C}_x\}_{x \in \mathcal{D}}$ (the construction does not depend this

choice). Define the birth morphism via the formula

$$\begin{aligned} b_x : \mathbb{1} &\rightarrow \mathbb{C}_x \otimes \mathbb{C}_{-x} \\ v_0 &\mapsto v_x \otimes v_{-x} \end{aligned} \tag{5.87}$$

We do *not* define the death morphism via the obvious formula

$$\begin{aligned} d_x : \mathbb{C}_{-x} \otimes \mathbb{C}_x &\rightarrow \mathbb{1} \\ v_{-x} \otimes v_x &\mapsto v_0 \end{aligned} \tag{5.88}$$

Instead we are obligated to enforce the rigidity conditions in equation (4.35). Consider the first sequence of maps in equation (4.35) (the second sequence is similar and provides identical information). For a simple object \mathbb{C}_x the sequence (which must equal id_x) is:

$$\begin{aligned} v_x &\xrightarrow{l_x^{-1}} v_0 \otimes v_x \xrightarrow{b_x \otimes \text{id}_x} (v_x \otimes v_{-x}) \otimes v_x \xrightarrow{a_{x,-x,x}} \\ &[a_{x,-x,x}] \cdot v_x \otimes (v_{-x} \otimes v_x) \xrightarrow{\text{id}_x \otimes d_x} \\ &[a_{x,-x,x} \cdot d_x] \cdot v_x \otimes v_0 \xrightarrow{r_V} [a_{x,-x,x} \cdot d_x] \cdot v_x \end{aligned} \tag{5.89}$$

this implies that

$$a_{x,-x,x} \cdot d_x = 1 \tag{5.90}$$

i.e.

$$\exp(2\pi i h(x, -x, x)) \cdot d_x = 1 \tag{5.91}$$

Hence we define the death morphism by the formula

$$d_x : \mathbb{C}_{-x} \otimes \mathbb{C}x \rightarrow \mathbb{1} \quad (5.92)$$

$$v_{-x} \otimes v_x \mapsto \exp(-2\pi i h(x, -x, x))v_0 \quad (5.93)$$

Collecting these facts, we have proven:

Proposition 5.94. *The group category $\mathcal{C}_{(\mathcal{D}, q)}(h, s)$ extended by the above twisting and rigidity is a finitely-semisimple ribbon category.*

Quantum dimension

The quantum dimension is defined by equation (4.39). We reuse the following lemma several times in the sequel:

Lemma 5.95. *Let \mathbb{C}_x be a simple object in $\mathcal{C}_{(\mathcal{D}, q)}(h, s)$. Then the map*

$$d_x \circ c_{x, -x} \circ (\theta_x \otimes id_{-x}) : \mathbb{C}_x \otimes \mathbb{C}_{-x} \cong \mathbb{1} \rightarrow \mathbb{1} \quad (5.96)$$

is just multiplication by 1.

Proof. This is a calculation (with a fairly tricky point that has confused the author more than once). θ_x is multiplication by the coefficient $\exp(2\pi i q(x))$. The braiding $c_{x, -x}$ is multiplication by $\exp(2\pi i s(x, -x))$, and the death operator d_x is multiplication by $\exp(-2\pi i h(x, -x, x))$. Hence the total coefficient is just $\exp[2\pi i (q(x) + s(x, -x) - h(x, -x, x))]$. In terms of ordered generators for \mathcal{D} we write $x = \sum_i a_i 1_i$. The tricky point is that it is *not* true that $-x = \sum_i (-a_i) 1_i$. In view of the commentary above equations (5.53) and (5.54)

we see instead that $-x = \sum_i (n_i - a_i) 1_i$, so

$$\begin{aligned}
q(x) + s(x, -x) &= \sum_{i < j} a_i a_j b_{ij} + \sum_i a_i a_i q_i \\
&+ \sum_{i < j} a_i (n_j - a_j) b_{ij} + \sum_i a_i (n_i - a_i) q_i = \\
&\sum_{i < j} a_i n_j b_{ij} + \sum_i a_i n_i q_i \quad (5.97)
\end{aligned}$$

However $\sum_{i < j} a_i n_j b_{ij} = \sum_{i < j} b(a_i 1_i, n_j 1_j) = \sum_{i < j} b(a_i 1_i, 0) = 0$. Hence we are left with

$$q(x) + s(x, -x) = \sum_i a_i n_i q_i \quad (5.98)$$

The death operator gives

$$h(x, -x, x) = \sum_i n_i a_i q_i \quad (5.99)$$

so

$$q(x) + s(x, -x) - h(x, -x, x) = \sum_i a_i n_i q_i - \sum_i a_i n_i q_i = 0 \quad (5.100)$$

Taking the exponent we get that the map is just multiplication by 1. \square

This easily implies the following (note: this result has nothing to do with the fact that the simple objects are 1-dimensional vector spaces \mathbb{C}_x ; the quantum dimension is not related):

Corollary 5.101. *The simple objects \mathbb{C}_x in $\mathcal{C}_{(\mathcal{D}, q)}(h, s)$ all have quantum dimension $\dim_q(\mathbb{C}_x) = 1$.*

Modular tensor category

In light of proposition (5.94) we only need to mention the rank \mathcal{D} and verify that the S matrix is invertible. Then we will have a modular tensor category.

The rank is

$$\mathcal{D} = \sqrt{\sum_{x \in \mathcal{D}} (\dim_q(\mathbb{C}_x))^2} = \sqrt{|\mathcal{D}|} \quad (5.102)$$

The coefficients of the S matrix are determined by equation (4.51). Recall that the quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ induces a bilinear form $b : \mathcal{D} \otimes \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$. A quick calculation using equation (5.54) shows that

$$S_{x,y} = \exp(2\pi i b(-x, y)) = \exp(-2\pi i b(x, y)) \quad (5.103)$$

This proves:

Theorem 5.104. *The group category $\mathcal{C}_{(\mathcal{D},q)}(h, s)$ extended with the twist and rigidity structure defined above is a modular tensor category iff the quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ is a refinement of a bilinear form $b : \mathcal{D} \otimes \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ such that the matrix $S_{x,y} = \exp(-2\pi i b(x, y))$ is invertible.*

We believe that the following proposition is true for all finite abelian groups, but we have only been able to prove it for cyclic groups:

Proposition 5.105. *Let \mathcal{D} be a cyclic group. Then the matrix $S_{x,y} = \exp(-2\pi i b(x, y))$ is invertible iff $b : \mathcal{D} \otimes \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ is non-degenerate.*

Proof. If b is degenerate then the matrix $b(x, y)$ has two rows consisting of zeros: the top row (since $b(0, y) = 0$) and another row $b(x, y) = 0$ for some $x \neq 0$. Hence the matrix

$$S_{x,y} = \exp(-2\pi i b(x, y)) \quad (5.106)$$

has two rows filled with 1's, hence $S_{x,y}$ is not invertible.

Conversely, suppose that b is non-degenerate. Let 1 be a generator for the cyclic group \mathcal{D} of order n , and define $X := \exp(-2\pi ib(1, 1))$. Then for integers $k, l = 0, 1, 2, \dots, n - 1$ we have the S -matrix

$$S_{k,l} := X^{kl} \tag{5.107}$$

A *Vandermonde determinant* is a determinant of a matrix of the form

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \dots \\ 1 & x_2 & x_2^2 & x_2^3 & \dots \\ 1 & x_3 & x_3^2 & x_3^3 & \dots \\ \vdots & & & & \end{pmatrix} \tag{5.108}$$

It is well-known that the determinant of this matrix is just

$$\prod_{0 \leq k < l \leq n-1} (x_l - x_k) \tag{5.109}$$

The S -matrix is of the Vandermonde form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & X & X^2 & X^3 & \dots \\ 1 & X^2 & X^4 & X^6 & \dots \\ 1 & X^3 & X^6 & X^9 & \dots \\ \vdots & & & & \end{pmatrix} \tag{5.110}$$

Since for non-degenerate b we have that $X^k \neq X^l$ when $k \neq l$ we see that the determinant of S is non-zero. □

Chapter 6

Main Theorem

6.1 Introduction

The goal of this chapter is to provide a correspondence between the toral (non-spin) Chern-Simons theories classified by Belov and Moore (see chapter (3)) and the group categories described in chapters (4) and (5). We achieve this by showing that the respective projective representations of the mapping class group ¹ are isomorphic.

Let Σ be a *closed* surface. The toral Chern-Simons projective representation of $\text{MCG}(\Sigma)$ factors through the symplectic group $\text{Sp}(2g, \mathbb{Z})$. This is explicitly given in equations (3.96), (3.97), and (3.98). The bulk of the work in this chapter concerns deriving the projective representation of $\text{MCG}(\Sigma)$ induced from $\mathcal{C}_{(\mathcal{D}, q)}(h, s)$ using surgery. The main work involves converting a Heegaard decomposition into a surgery presentation.

¹we restrict ourselves to closed surfaces

6.2 Projective representation of $\text{MCG}(\Sigma)$ from

$$\mathcal{C}_{(\mathcal{D},q)}(h, s)$$

As a first step we outline briefly some standard constructions from low-dimensional topology (see, for example, [PS96]).

Presentation of the mapping class group via Dehn-Lickorish twists

Since we wish to consider the group $\text{MCG}(\Sigma)$ we require an efficient presentation for it. It is well known that $\text{MCG}(\Sigma)$ is generated by compositions of Dehn twists around simple closed curves (see, for example, [FM07]). We use the standard “turn left” Dehn twist convention as depicted in figure (4.10). We note that “turn left” makes sense independent of any choice of orientation of the curves.

It is equally well known that for a closed surface Σ of genus g it suffices to consider only Dehn twists along the $3g - 1$ *Lickorish* generators depicted in figure (6.1). In what follows we will limit our study to the Lickorish generators.

Motivation

It was mentioned in chapter (4) (and studied thoroughly in [Tur94]) that a modular tensor category associates to any oriented 3-manifold X with boundary $\partial X = -\Sigma_- \sqcup \Sigma_+$ an operator

$$\tau(X) : \mathcal{F}(\Sigma_-) \rightarrow \mathcal{F}(\Sigma_+) \tag{6.1}$$

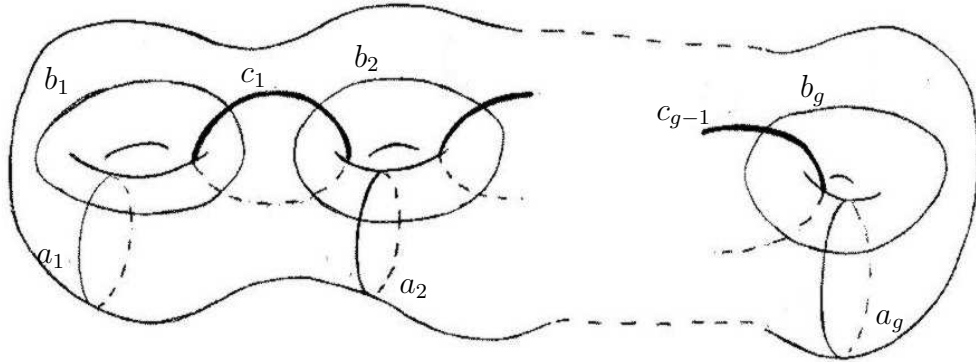


Figure 6.1: The $3g - 1$ Lickorish generators.

In general X needs to be endowed with some *extended structure* in order to construct a theory free from gluing anomalies. The boundary surfaces Σ_- and Σ_+ must be endowed with some extra structure (a parameterization here) to *construct* a theory at all. However, if we are satisfied with a TQFT *with anomaly* then the parameterization is irrelevant, and for an *anomaly-free* TQFT the dependence on parameterization is very weak (see chapter (2)). The matrix elements of $\tau(X)$ are defined by first “capping off” Σ_- and Σ_+ with the standard handlebodies H_{g_-} and $\overline{H_{g_+}}$, respectively.² We then choose a coloring for the embedded ribbon graphs R_{g_-} and $\overline{R_{g_+}}$. This gives a closed 3-manifold \tilde{X} with colored embedded ribbons. The matrix element (corresponding to the chosen coloring) is defined to be the 3-manifold invariant $\tau(\tilde{X}) \in \mathbb{C}$. Varying over all choices of coloring gives all of the matrix elements of the operator

$$\tau(X) : \mathcal{F}(\Sigma_-) \rightarrow \mathcal{F}(\Sigma_+) \quad (6.2)$$

In particular recall that this procedure provides a (projective) repre-

²using the parameterizations

sentation of the mapping class group for any surface Σ of genus g equipped with a parameterization $\phi : \partial H_g \rightarrow \Sigma$. We start by considering the cylinder $\Sigma \times I$ where both boundary components $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ have the same parameterization ϕ .

Then given an isotopy class of diffeomorphisms $[f] \in \text{MCG}(\Sigma)$ pick a representative diffeomorphism $f : \Sigma \rightarrow \Sigma$. Then *alter* the parameterization of the boundary component $\Sigma \times \{0\}$ to be

$$f \circ \phi \tag{6.3}$$

Denote $\Sigma \times I$ (with the altered parameterization of $\Sigma \times \{0\}$) by X_f . Then the operator

$$\tau(X_f) : \mathcal{F}(\Sigma) \rightarrow \mathcal{F}(\Sigma) \tag{6.4}$$

defines a projective representation of $\text{MCG}(\Sigma)$.

Converting Heegaard decomposition to integer surgery presentation

We just saw that in order to study the projective action of the $\text{MCG}(\Sigma)$ we cap off the 3-manifold X_f with standard handlebodies to form \tilde{X}_f . However, since $\Sigma \times I$ deformation retracts onto Σ by collapsing the interval I , we can view the closed manifold \tilde{X}_f as two solid handlebodies glued along f . This provides a Heegaard decomposition for \tilde{X}_f (however the standard handlebodies contain the embedded ribbon graphs R_g and \overline{R}_g , respectively).

To find the matrix elements we are required to calculate the 3-manifold invariant $\tau(\tilde{X}_f)$. However, the machinery described in chapter (4) relies on

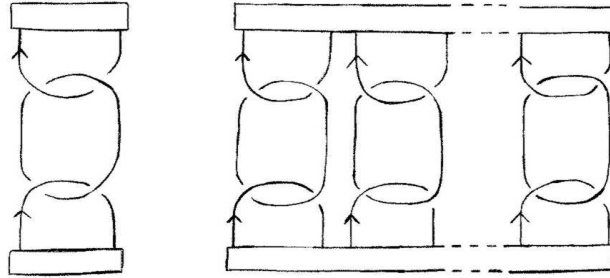


Figure 6.2: A link diagram in S^3 that reproduces the identity map that glues two genus 1 standard handlebodies together (left) or more generally two genus g standard handlebodies (right). The bottom component is R_g and the top component is \overline{R}_g . The middle link(s) encode the surgery.

an integer surgery presentation instead. Hence we are left with the task of converting a Heegaard decomposition into a surgery presentation. Our task is greatly simplified since $\text{MCG}(\Sigma)$ is generated by the Lickorish generators.

First suppose $f = \text{id}$ (so we have two genus g standard handlebodies glued together along the identity boundary diffeomorphism). We want to obtain this manifold from integer surgery along links in S^3 . In genus 1 this is straightforward and already described in chapter (4). Two solid tori glued together along the identity boundary diffeomorphism is just $S^2 \times S^1$. This can be obtained from S^3 (see figure (4.12)) by a single torus switch, i.e. a 0-framed surgery (see figure (4.11)).

If we remember to place the ribbon graphs R_g and \overline{R}_g into the handlebodies then we obtain a surgery presentation in S^3 as in figure (6.2) (left side). Note that the ribbon graphs R_g and \overline{R}_g do *not* participate in the surgery.

Now consider a Dehn twist along one of the Lickorish simple closed curves in figure (6.1). There is a surgery that is equivalent to performing this Dehn twist. The trick is sketched in [PS96] on pg. 85. The appropriate surgery

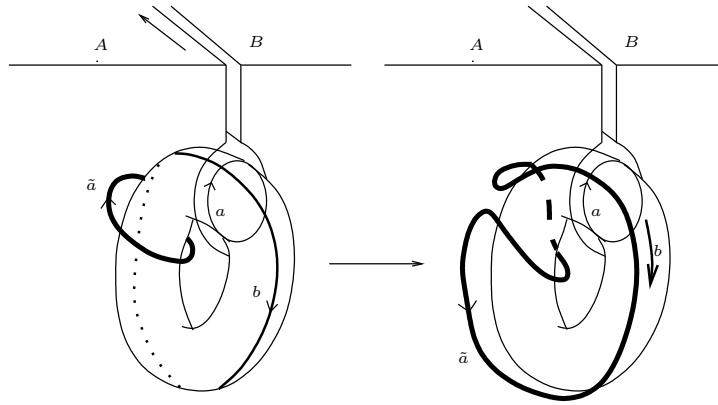


Figure 6.3: A cross-section of the handlebody. The Dehn twist takes place on a simple closed curve (not shown) separating A and B . The curve is pushed into the handlebody, slicing it. It is then thickened up to a solid torus, and then is drilled out. This leaves a torus-shaped hole (not drawn) in the handlebody. The region labelled A is rotated past B one full turn (making sure that any necessary deformation is restricted to the torus-shaped hole). The solid torus is then glued back in. The curve \tilde{b} is not shown.

entails the following steps. First push the curve slightly into the handlebody H_g . As the curve is pushed let it slice the handlebody (see figure (6.3)). Now thicken up the curve to a solid torus and drill it out (this leaves a torus-shaped “hole”). Next draw some markings a and b on the solid torus and matching markings \tilde{a} and \tilde{b} on the complementary hole (choose orientations as in figure (4.9)). Perform the Dehn twist by sliding A past B one complete revolution and then regluing (we can confine any necessary stretching to the torus-shaped hole). Now glue the solid torus back in. This produces the following identifications:

$$\begin{aligned}\tilde{a} &= a - b \\ \tilde{b} &= b\end{aligned}\tag{6.5}$$

This procedure can be viewed equivalently as *not* stretching in the hole, but rather stretching the solid torus in the opposite direction and gluing it back in. In other words we can equivalently solve for a and b in terms of \tilde{a} and \tilde{b} to obtain

$$\begin{aligned} a &= \tilde{a} + \tilde{b} \\ b &= \tilde{b} \end{aligned} \tag{6.6}$$

which is just the surgery matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{6.7}$$

i.e. a 1-framed surgery (as in example (4.72)). This shows that we can perform a Dehn twist along a simple closed curve as in figure (6.1) by replacing it with a 1-framed surgery along the same simple closed curve. Let us exploit this by providing surgery presentations for the Lickorish generators $\{a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_{g-1}\}$ as in figures (6.4), (6.5), and (6.6).

Representation of Lickorish generators from $\mathcal{C}_{(\mathcal{D},q)}(h, s)$

Given the simplistic fusion rules for the simple objects of $\mathcal{C}_{(\mathcal{D},q)}(h, s)$ (that were specified in chapter (5)) it is easy to see that for $x_1, \dots, x_g \in \mathcal{D}$ the following tensor product is 1-dimensional:

$$\mathbb{C}_{x_1} \otimes \mathbb{C}_{-x_1} \otimes \dots \otimes \mathbb{C}_{x_g} \otimes \mathbb{C}_{-x_g} \cong \mathbb{1} \tag{6.8}$$

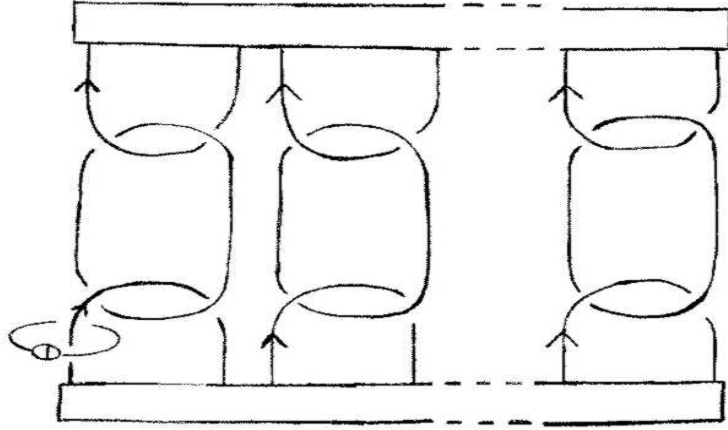


Figure 6.4: A link diagram in S^3 that reproduces the Dehn twist along a_1 as in figure (6.1). The bottom component is R_g and the top component is \overline{R}_g . The unoriented links encode the surgery.

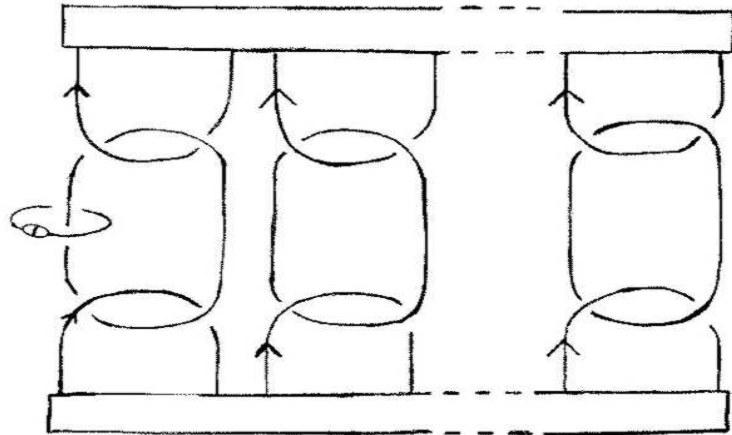


Figure 6.5: A link diagram in S^3 that reproduces the Dehn twist along b_1 as in figure (6.1). The bottom component is R_g and the top component is \overline{R}_g . The unoriented links encode the surgery.

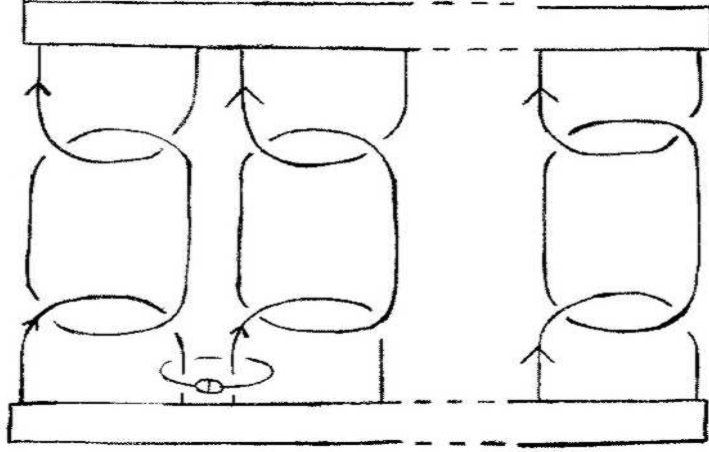


Figure 6.6: A link diagram in S^3 that reproduces the Dehn twist along c_1 as in figure (6.1). The bottom component is R_g and the top component is \overline{R}_g . The unoriented links encode the surgery.

Since $\text{Hom}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$ we see that

$$\text{Hom}(\mathbb{1}, \mathbb{C}_{x_1} \otimes \mathbb{C}_{-x_1} \otimes \cdots \otimes \mathbb{C}_{x_g} \otimes \mathbb{C}_{-x_g}) \cong \mathbb{C} \quad (6.9)$$

is 1-dimensional.

Therefore given a coloring $x_1, \dots, x_g \in \mathcal{D}$ for the ribbons in figure (4.13) (embedded in H_g) the coloring of the coupon $\in \text{Hom}(\mathbb{1}, \mathbb{C}_{x_1} \otimes \mathbb{C}_{-x_1} \otimes \cdots \otimes \mathbb{C}_{x_g} \otimes \mathbb{C}_{-x_g})$ is essentially unique (up to a complex constant). In terms of a basis $v_x \in \mathbb{C}_x$ for the simple objects let us (for example) color the coupon with the linear map

$$v_0 \mapsto v_{x_1} \otimes v_{-x_1} \otimes \cdots \otimes v_{x_g} \otimes v_{-x_g} \quad (6.10)$$

However, since the associativity isomorphisms are non-trivial we should be

careful with parenthesis (we choose the convention to group from the left):

$$v_0 \mapsto (\cdots (([v_{x_1} \otimes v_{-x_1}] \otimes [v_{x_2} \otimes v_{-x_2}]) \otimes [v_{x_3} \otimes v_{-x_3}]) \otimes \cdots \otimes [v_{x_g} \otimes v_{-x_g}]) \quad (6.11)$$

Similarly, for the handlebody \overline{H}_g the space

$$\text{Hom}(\mathbb{C}_{x_1} \otimes \mathbb{C}_{-x_1} \otimes \cdots \otimes \mathbb{C}_{x_g} \otimes \mathbb{C}_{-x_g}, \mathbb{1}) \cong \mathbb{C} \quad (6.12)$$

is 1-dimensional (the associativity parenthesis have been omitted to avoid confusion). Given a coloring $x_1, \dots, x_g \in \mathcal{D}$ for the ribbons in figure (4.14) we color the coupon with the linear morphism (for example)

$$(\cdots (([v_{x_1} \otimes v_{-x_1}] \otimes [v_{x_2} \otimes v_{-x_2}]) \otimes [v_{x_3} \otimes v_{-x_3}]) \otimes \cdots \otimes [v_{x_g} \otimes v_{-x_g}]) \mapsto v_0 \quad (6.13)$$

The computed matrix elements depend on the choices made above, however it is easy to see (see equation (4.90)) that all choices made above are equivalent to choosing a basis for the Hilbert space $\mathcal{F}(\Sigma)$. Hence the operator is well-defined independent of these choices.

The identity diffeomorphism $\text{id} : \Sigma \rightarrow \Sigma$ (sanity check)

Let us proceed to calculate the matrix corresponding to the identity diffeomorphism

$$\text{id} : \Sigma \rightarrow \Sigma \quad (6.14)$$

The surgery presentation for this is given in figure (6.2). In genus g the different vertical braid sections do not interact (on the right side of figure (6.2)), hence we restrict ourselves to genus 1 and the genus g calculation will be g

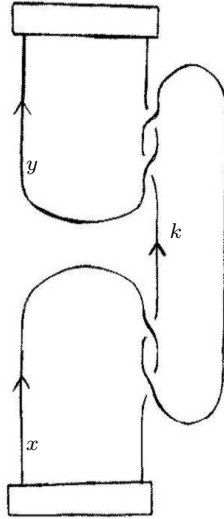


Figure 6.7: Surgery presentation of identity diffeomorphism $\text{id} : \Sigma \rightarrow \Sigma$ in genus 1. The surgery is performed on the link component colored by \mathbb{C}_k .

copies of the genus 1 calculation tensored together. Consult figure (6.7). It is understood that $x \in \mathcal{D}$ and $y \in \mathcal{D}$ are fixed, and $k \in \mathcal{D}$ is summed over since that component performs the surgery.

We note that we are required to explicitly write the associativity maps since they are nontrivial (see chapter (5)). However, we shall see shortly that they cancel each other (this is only true because the category is abelian), hence we will drop the explicit associativity maps quickly.

Also we recall lemma (5.95). When we annihilate a simple object \mathbb{C}_x and its dual \mathbb{C}_{-x} we do not bother to write the map $d_x \circ c_{x,-x} \circ (\theta_x \otimes \text{id}_{-x})$ since it is trivial.

Following the diagram from the bottom to the top we compute

$$\begin{aligned}
v_0 &\mapsto v_x \otimes v_{-x} & (6.15) \\
&\mapsto (v_x \otimes v_{-x}) \otimes (v_k \otimes v_{-k}) \\
&\mapsto [a_{x,-x,k-k}]v_x \otimes (v_{-x} \otimes (v_k \otimes v_{-k})) \\
&\mapsto [a_{x,-x,k-k}][a_{-x,k,-k}^{-1}]v_x \otimes ((v_{-x} \otimes v_k) \otimes v_{-k}) \\
&\mapsto [a_{x,-x,k-k}][a_{-x,k,-k}^{-1}][c_{k,-x}^{-1}]v_x \otimes ((v_k \otimes v_{-x}) \otimes v_{-k}) \\
&\mapsto [a_{x,-x,k-k}][a_{-x,k,-k}^{-1}][c_{k,-x}^{-1}][c_{-x,k}^{-1}]v_x \otimes ((v_{-x} \otimes v_k) \otimes v_{-k}) \\
&\mapsto [a_{x,-x,k-k}][a_{-x,k,-k}^{-1}][c_{k,-x}^{-1}][c_{-x,k}^{-1}][a_{-x,k,-k}]v_x \otimes (v_{-x} \otimes (v_k \otimes v_{-k})) \\
&\mapsto [a_{x,-x,k-k}][a_{-x,k,-k}^{-1}][c_{k,-x}^{-1}][c_{-x,k}^{-1}][a_{-x,k,-k}][a_{x,-x,k-k}^{-1}](v_x \otimes v_{-x}) \otimes (v_k \otimes v_{-k})
\end{aligned}$$

Clearly the associativity coefficients cancel each other. Annihilating $\mathbb{C}_x \otimes \mathbb{C}_{-x}$ we obtain

$$\mapsto [c_{k,-x}^{-1}][c_{-x,k}^{-1}]v_k \otimes v_{-k} \quad (6.16)$$

It is easy to convince ourselves that the associativity maps are always going to appear in cancelling pairs, hence we omit them from here on to simplify notation. Note that, in principle, the associativity maps must be included. Continuing up the diagram, there is a birth of $\mathbb{C}_y \otimes \mathbb{C}_{-y}$:

$$\begin{aligned}
&\mapsto [c_{k,-x}^{-1}][c_{-x,k}^{-1}](v_y \otimes v_{-y}) \otimes (v_k \otimes v_{-k}) & (6.17) \\
&\mapsto [c_{k,-x}^{-1}][c_{-x,k}^{-1}][c_{-y,k}]v_y \otimes v_k \otimes v_{-y} \otimes v_{-k} \\
&\mapsto [c_{k,-x}^{-1}][c_{-x,k}^{-1}][c_{-y,k}][c_{k,-y}](v_y \otimes v_{-y}) \otimes (v_k \otimes v_{-k})
\end{aligned}$$

Annihilate $\mathbb{C}_k \otimes \mathbb{C}_{-k}$, then apply the map in equation (6.13) to annihilate

$$\mathbb{C}_y \otimes \mathbb{C}_{-y}: \quad \mapsto [c_{k,-x}^{-1}][c_{-x,k}^{-1}][c_{-y,k}][c_{k,-y}]v_0 \quad (6.18)$$

Hence the ribbon invariant $F(L \cup \Omega)$ is just

$$[c_{k,-x}^{-1}][c_{-x,k}^{-1}][c_{-y,k}][c_{k,-y}] \quad (6.19)$$

To calculate the 3-manifold invariant we use equation (4.77). We note that the quantum dimension $\dim_q(\mathbb{C}_x) = 1$ for all simple objects, hence we omit the dimension factor. The L surgery link is the one colored by \mathbb{C}_k , and the fixed ribbon Ω is the two component ribbon graph colored by \mathbb{C}_x and \mathbb{C}_y .

Summing over colorings is the same as summing over $k \in \mathcal{D}$. So we have

$$\tau(X_{\text{id}}) = (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_{k \in \mathcal{D}} F(L \cup \Omega) \quad (6.20)$$

We calculate using equation (5.54)

$$\begin{aligned} [c_{k,-x}^{-1}][c_{-x,k}^{-1}] &= \exp(2\pi i b(x, k)) \\ [c_{-y,k}][c_{k,-y}] &= \exp(-2\pi i b(y, k)) \end{aligned} \quad (6.21)$$

hence

$$\tau(X_{\text{id}}) = (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_{k \in \mathcal{D}} \exp(2\pi i b(x, k)) \exp(-2\pi i b(y, k)) \quad (6.22)$$

Using the bilinearity and symmetry of b this becomes

$$\tau(X_{\text{id}}) = (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_{k \in \mathcal{D}} \exp(2\pi i b(x - y, k)) \quad (6.23)$$

Now we appeal to lemma (6.27) (see below). The 3-manifold invariant becomes

$$\tau(X_{\text{id}}) = (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \mathcal{D}^2 \delta_{x,y} \quad (6.24)$$

The signature of the linking matrix for L is just $\sigma(L) = 0$, and in genus $g = 1$ we have $m = 1$ component of L . So the 3-manifold invariant is

$$\tau(X_{\text{id}}) = \mathcal{D}^{-2} \mathcal{D}^2 \delta_{x,y} = \delta_{x,y} \quad (6.25)$$

as we expect for the identity diffeomorphism $\text{id} : \Sigma \rightarrow \Sigma$.

In genus g (see right side of figure (6.2)) we have $m = g$ components of L (it is still true that $\sigma(L) = 0$) and the 3-manifold invariant becomes g copies of $\mathcal{D}^2 \delta_{x,y}$ tensored together (the normalization must be considered separately):

$$\begin{aligned} \tau(X_{\text{id}}) &= \mathcal{D}^{-g-1} \mathcal{D}^{2g} \delta_{x_1, y_1} \cdots \delta_{x_g, y_g} \\ &= \mathcal{D}^{g-1} \delta_{x_1, y_1} \cdots \delta_{x_g, y_g} \end{aligned} \quad (6.26)$$

The projective factor in front is a symptom that we only have a projective representation of $\text{MCG}(\Sigma)$.

Lemma 6.27.

$$\sum_{k \in \mathcal{D}} \exp(2\pi i b(g, k)) = \mathcal{D}^2 \delta_{g,0} \quad (6.28)$$

Proof. Clearly if $g = 0$ then the LHS will just be $|\mathcal{D}|$, i.e. \mathcal{D}^2 for the special case $\mathcal{C}_{(\mathcal{D}, q)}(h, s)$ since $\dim_q \mathbb{C}_x = 1$ for all simple objects.

Suppose $g \neq 0$. In terms of generators $1_1, \dots, 1_p$ for the group \mathcal{D} write

$g = \sum_i g_i 1_i$ and write any arbitrary element $k = \sum_i k_i 1_i$. The sum becomes

$$\begin{aligned} \sum_{k \in \mathcal{D}} \exp(2\pi i \sum_{i,j} g_i k_j b_{ij}) = \\ \sum_{k_1=0}^{k_1=n_1-1} \cdots \sum_{k_p=0}^{k_p=n_p-1} \prod_i \exp(2\pi i g_i k_1 b_{i1}) \dots \exp(2\pi i g_i k_p b_{ip}) \end{aligned} \quad (6.29)$$

Consider the last sum by itself. We intend to show that this vanishes.

$$\sum_{k_p=0}^{k_p=n_p-1} \prod_i \exp(2\pi i g_i k_1 b_{i1}) \dots \exp(2\pi i g_i k_p b_{ip}) \quad (6.30)$$

This can be written

$$\begin{aligned} \sum_{k_p=0}^{k_p=n_p-1} \prod_i \exp(2\pi i g_i k_1 b_{i1}) \dots \exp(2\pi i g_i k_{p-1} b_{i,(p-1)}) \prod_i \exp(2\pi i g_i k_p b_{ip}) = \\ \prod_i \exp(2\pi i g_i k_1 b_{i1}) \dots \exp(2\pi i g_i k_{p-1} b_{i,(p-1)}) \sum_{k_p=0}^{k_p=n_p-1} \prod_i \exp(2\pi i g_i k_p b_{ip}) \end{aligned} \quad (6.31)$$

Again restricting attention to the last sum this is

$$\sum_{k_p=0}^{k_p=n_p-1} \left[\prod_i \exp(2\pi i g_i b_{ip}) \right]^{k_p} \quad (6.32)$$

However $n_p b_{ip} = b(1_i, n_p 1_p) = b(1_i, 0) = 0$ so we see that $[\prod_i \exp(2\pi i g_i b_{ip})]$ is an n_p -th root of unity. Hence the terms $[\prod_i \exp(2\pi i g_i b_{ip})]^0, [\prod_i \exp(2\pi i g_i b_{ip})]^1, \dots, [\prod_i \exp(2\pi i g_i b_{ip})]^{n_p}$ will be symmetrically distributed around the unit circle, so the sum will be 0. \square

Dehn twist along a_1

Let us proceed to calculate the matrix corresponding to the Dehn twist along the curve a_1 depicted in figure (6.1). Dehn twists along the other a_i curves are similar. Denote the Dehn twist diffeomorphism as

$$T_{a_i} : \Sigma \rightarrow \Sigma \tag{6.33}$$

The surgery presentation for this is given in figure (6.4). In genus g the different vertical braid sections again do not interact (see figure (6.4)), hence we restrict ourselves to genus 1 since the genus g calculation can be recovered by tensoring the genus 1 calculation here with $g - 1$ copies of the genus 1 id calculation as in equation (6.24) (the normalization must be considered separately). Consult figure (6.8). It is understood that $x \in \mathcal{D}$ and $y \in \mathcal{D}$ are fixed, and $k, l \in \mathcal{D}$ are summed over since those components perform the surgery.

In genus 1 we write

$$T_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{MCG}(\text{torus}) \tag{6.34}$$

Again, as in the last calculation we drop the explicit associativity maps since they cancel each other. Technically they should be written, however.

Also again recall lemma (5.95). When we annihilate a simple object \mathbb{C}_x and its dual \mathbb{C}_{-x} we do not bother to write the map $d_x \circ c_{x,-x} \circ (\theta_x \otimes \text{id}_{-x})$ since it is trivial.

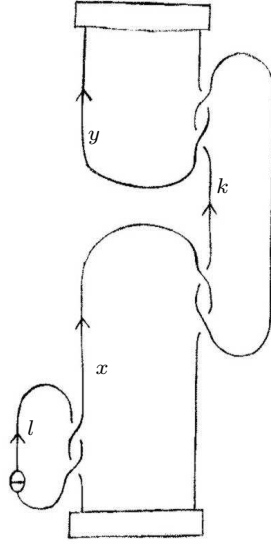


Figure 6.8: Surgery presentation of Dehn twist along a in genus 1. The surgery is performed on the link components colored by \mathbb{C}_k and \mathbb{C}_l .

Following the diagram from the bottom to the top we compute

$$v_0 \mapsto v_x \otimes v_{-x} \tag{6.35}$$

$$\mapsto (v_l \otimes v_{-l}) \otimes (v_x \otimes v_{-x})$$

$$\mapsto [\theta_l] v_l \otimes v_{-l} \otimes v_x \otimes v_{-x}$$

$$\mapsto [\theta_l][c_{-l,x}] v_l \otimes v_x \otimes v_{-l} \otimes v_{-x}$$

$$\mapsto [\theta_l][c_{-l,x}][c_{x,-l}] (v_l \otimes v_{-l}) \otimes (v_x \otimes v_{-x})$$

$$\tag{6.36}$$

Now annihilating $\mathbb{C}_l \otimes \mathbb{C}_{-l}$ gives

$$\mapsto [\theta_l][c_{-l,x}][c_{x,-l}] v_x \otimes v_{-x}$$

The remainder of the calculation proceeds exactly as for the genus 1 id braid used to calculate equation (6.24). This implies that the ribbon invariant is

$$F(L \cup \Omega) = [c_{k,-x}^{-1}][c_{-x,k}^{-1}][c_{-y,k}][c_{k,-y}][\theta_l][c_{-l,x}][c_{x,-l}] \quad (6.37)$$

Using equation (5.54) we compute

$$\begin{aligned} [c_{k,-x}^{-1}][c_{-x,k}^{-1}] &= \exp(2\pi ib(x, k)) \\ [c_{-y,k}][c_{k,-y}] &= \exp(-2\pi ib(y, k)) \\ [c_{-l,x}][c_{x,-l}] &= \exp(-2\pi ib(l, x)) \\ [\theta_l] &= \exp(2\pi iq(l)) \end{aligned} \quad (6.38)$$

which implies that the 3-manifold invariant $\tau(X_{T_a})$ given by equation (4.77) is

$$\begin{aligned} \tau(X_{T_a}) &= (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \\ &\quad \sum_{k,l \in \mathcal{D}} \exp(2\pi ib(x, k)) \exp(-2\pi ib(y, k)) \exp(-2\pi ib(l, x)) \exp(2\pi iq(l)) \end{aligned} \quad (6.39)$$

Breaking the sum up

$$\sum_l \exp(-2\pi ib(l, x)) \exp(2\pi iq(l)) \sum_k \exp(2\pi ib(x, k)) \exp(-2\pi ib(y, k)) \quad (6.40)$$

But by lemma (6.27) the sum over k becomes $\mathcal{D}^2 \delta_{xy}$. Hence the 3-manifold invariant is

$$\tau(X_{T_a}) = (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \mathcal{D}^2 \delta_{xy} \sum_l \exp(-2\pi ib(l, x)) \exp(2\pi iq(l)) \quad (6.41)$$

Now we use the properties of the bilinear form $-b(l, x) = b(l, -x) = q(l - x) - q(-x) - q(l)$ and substitute to obtain

$$\tau(X_{T_a}) = (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \mathcal{D}^2 \delta_{xy} \exp(-2\pi i q(x)) \sum_l \exp(2\pi i q(l-x)) \quad (6.42)$$

We have used the fact that $q(-x) = q(x)$ for a pure quadratic form. The last sum is just p_+ from chapter (4), so the 3-manifold invariant is

$$\tau(X_{T_a}) = p_+(p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \cdot \mathcal{D}^2 \exp(-2\pi i q(x)) \delta_{xy} \quad (6.43)$$

In genus 1 we see that the signature of the linking matrix for L is just $\sigma(L) = 1$ (the component colored by \mathbb{C}_l has a 1-framing, the component colored by \mathbb{C}_k has a zero framing, and the components are not linked with each other). The number of components of L is $m = 2$. Hence the 3-manifold invariant is

$$\tau(X_{T_a}) = \exp(-2\pi i q(x)) \delta_{xy} \quad (6.44)$$

where we have used the fact that $p_+ p_- = \mathcal{D}^2$.

In genus g (as in figure (6.4)) this computation is tensored with $g - 1$ genus 1 id calculations. We recall that from equation (6.24) each genus 1 id computation (without normalization) gives a factor of $\mathcal{D}^2 \delta_{x_i y_i}$. There are $m = g + 1$ link components, and the signature is still $\sigma(L) = 1$. Thus the 3-manifold invariant for the Dehn twist T_{a_i} is

$$\begin{aligned} \tau(X_{T_{a_i}}) &= p_+ p_- \mathcal{D}^{-1-(g+1)-1} \mathcal{D}^2 \mathcal{D}^{2(g-1)} \exp(-2\pi i q(x_i)) \delta_{x_1 y_1} \dots \delta_{x_g y_g} \quad (6.45) \\ &= \mathcal{D}^{g-1} \exp(-2\pi i q(x_i)) \delta_{x_1 y_1} \dots \delta_{x_g y_g} \end{aligned}$$

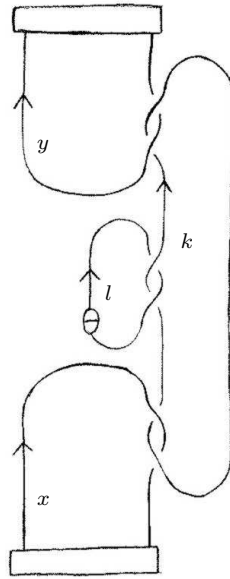


Figure 6.9: Surgery presentation of Dehn twist along b in genus 1. The surgery is performed on the link components colored by \mathbb{C}_k and \mathbb{C}_l .

Dehn twist along b_1

The computation for a Dehn twist along b_1 is nearly identical. Again we can restrict to genus 1 as in figure (6.9).

In genus 1 we write

$$T_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \text{MCG}(\text{torus}) \quad (6.46)$$

Rather than follow a similar tedious computation we skip to the result

$$F(L \cup \Omega) = [c_{k,-x}^{-1}][c_{-x,k}^{-1}][c_{-y,k}][c_{k,-y}][\theta_l][c_{-l,k}][c_{k,-l}] \quad (6.47)$$

Using equation (5.54) compute

$$\begin{aligned}
[c_{k,-x}^{-1}][c_{-x,k}^{-1}] &= \exp(2\pi ib(x, k)) \\
[c_{-y,k}][c_{k,-y}] &= \exp(-2\pi ib(y, k)) \\
[c_{-l,k}][c_{k,-l}] &= \exp(-2\pi ib(l, k)) \\
[\theta_l] &= \exp(2\pi iq(l))
\end{aligned} \tag{6.48}$$

which implies that the 3-manifold invariant $\tau(X_{T_b})$ given by equation (4.77) is

$$\begin{aligned}
\tau(X_{T_b}) &= (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \\
&\sum_{k,l \in \mathcal{D}} \exp(2\pi ib(x, k)) \exp(-2\pi ib(y, k)) \exp(-2\pi ib(l, k)) \exp(2\pi iq(l))
\end{aligned} \tag{6.49}$$

Summing over k and using lemma (6.27) this becomes

$$\tau(X_{T_b}) = (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_{l \in \mathcal{D}} \mathcal{D}^2 \exp(2\pi iq(l)) \delta_{x-y-l,0} \tag{6.50}$$

which is just

$$\tau(X_{T_b}) = (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \mathcal{D}^2 \exp(2\pi iq(x-y)) \tag{6.51}$$

In genus 1 there are $m = 2$ components of L . The \mathbb{C}_l -colored component has framing 1. The \mathbb{C}_k -colored component has framing 0. These two components have linking number -1 with respect to each other. Hence the linking matrix is

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \tag{6.52}$$

We see that $\det(B) = -1$, hence there is 1 positive and 1 negative eigenvalue.

So the signature is $\sigma(L) = 0$.

Thus, in genus 1 we see that

$$\tau(X_{T_b}) = \mathcal{D}^{-2-1} \mathcal{D}^2 \exp(2\pi i q(x - y)) = \frac{1}{\mathcal{D}} \exp(2\pi i q(x - y)) \quad (6.53)$$

In genus g if we perform a Dehn twist along b_i and tensor with $g - 1$ copies of the genus 1 id computation then we have $m = g + 1$ surgery link components as in figure (6.5). It is easy to verify that the signature remains $\sigma(L) = 0$. Thus the 3-manifold invariant is just

$$\begin{aligned} \tau(X_{T_{b_i}}) = \mathcal{D}^{-(g+1)-1} \mathcal{D}^2 \exp(2\pi i q(x_i - y_i)) \\ \mathcal{D}^{2(g-1)} \delta_{x_1 y_1} \dots \delta_{x_{i-1} y_{i-1}} \delta_{x_{i+1} y_{i+1}} \dots \delta_{x_g y_g} \end{aligned} \quad (6.54)$$

which is

$$\tau(X_{T_{b_i}}) = \mathcal{D}^{g-2} \exp(2\pi i q(x_i - y_i)) \delta_{x_1 y_1} \dots \delta_{x_{i-1} y_{i-1}} \delta_{x_{i+1} y_{i+1}} \dots \delta_{x_g y_g} \quad (6.55)$$

Dehn twist along c_1

The computation for a Dehn twist along c_1 is only slightly more involved. In this example there is no genus 1 case because two vertical braid sections interact (see figure (6.6)). Consider the genus 2 case as in figure (6.10). As usual we can consider the genus g case by tensoring with $g - 2$ copies of the genus 1 id computation as in equation (6.24) (the normalization must be

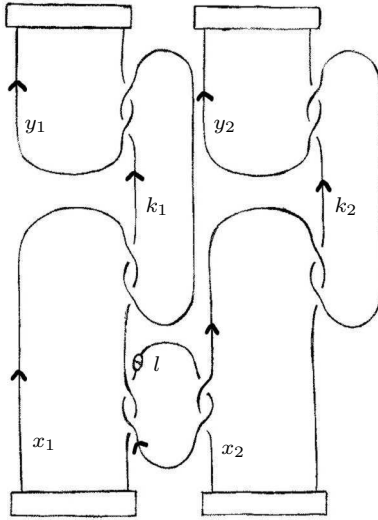


Figure 6.10: Surgery presentation of Dehn twist along c in genus 2. The surgery is performed on the link components colored by \mathbb{C}_{k_1} , \mathbb{C}_{k_2} , and \mathbb{C}_l .

considered separately). The case of a Dehn twist

$$T_{c_i} : \Sigma \rightarrow \Sigma \tag{6.56}$$

along an arbitrary c_i is similar. We drop the explicit associativity maps.

Following the diagram up we compute:

$$\begin{aligned}
v_0 &\mapsto (v_{x_1} \otimes v_{-x_1}) \otimes (v_{x_2} \otimes v_{-x_2}) & (6.57) \\
&\mapsto (v_{x_1} \otimes v_{-x_1}) \otimes (v_l \otimes v_{-l}) \otimes (v_{x_2} \otimes v_{-x_2}) \\
&\mapsto [c_{l,-x_1}^{-1}][c_{-l,x_2}]v_{x_1} \otimes v_l \otimes v_{-x_1} \otimes v_{x_2} \otimes v_{-l} \otimes v_{-x_2} \\
&\mapsto [c_{l,-x_1}^{-1}][c_{-x_1,l}^{-1}][c_{-l,x_2}][c_{x_2,-l}]v_{x_1} \otimes v_{-x_1} \otimes v_l \otimes v_{-l} \otimes v_{-x_2} \otimes v_{-x_2} \\
&\mapsto [c_{l,-x_1}^{-1}][c_{-x_1,l}^{-1}][c_{-l,x_2}][c_{x_2,-l}][\theta_l]v_{x_1} \otimes v_{-x_1} \otimes v_l \otimes v_{-l} \otimes v_{-x_2} \otimes v_{-x_2} \\
&\mapsto [c_{l,-x_1}^{-1}][c_{-x_1,l}^{-1}][c_{-l,x_2}][c_{x_2,-l}][\theta_l]v_{x_1} \otimes v_{-x_1} \otimes v_{-x_2} \otimes v_{-x_2} & (6.58)
\end{aligned}$$

where in the last line the pair $\mathbb{C}_l \otimes \mathbb{C}_{-l}$ has been annihilated.

From here the diagram proceeds as two copies of the genus 1 id computation. Hence (copying the results before equation (6.24)) we obtain that the ribbon graph invariant $F(L \cup \Omega)$ is

$$\begin{aligned}
&[c_{l,-x_1}^{-1}][c_{-x_1,l}^{-1}][c_{-l,x_2}][c_{x_2,-l}][\theta_l] \times \\
&\quad [c_{k_1,-x_1}^{-1}][c_{-x_1,k_1}^{-1}][c_{-y_1,k_1}][c_{k_1,-y_1}] \times \\
&\quad [c_{k_2,-x_2}^{-1}][c_{-x_2,k_2}^{-1}][c_{-y_2,k_2}][c_{k_2,-y_2}] & (6.59)
\end{aligned}$$

Writing this out using equation (5.54) this becomes

$$\begin{aligned}
&\exp(2\pi ib(l, x_1))\exp(2\pi ib(-l, x_2))\exp(2\pi iq(l)) \times \\
&\quad \exp(2\pi ib(k_1, x_1))\exp(-2\pi ib(y_1, k_1)) \times \\
&\quad \exp(2\pi ib(k_2, x_2))\exp(-2\pi ib(y_2, k_2)) & (6.60)
\end{aligned}$$

The 3-manifold invariant is calculated using equation (4.77):

$$\begin{aligned}
\tau(X_{T_c}) &= (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_{l, k_1, k_2} \\
&\quad \exp(2\pi i b(l, x_1)) \exp(2\pi i b(-l, x_2)) \exp(2\pi i q(l)) \times \\
&\quad \exp(2\pi i b(k_1, x_1)) \exp(-2\pi i b(y_1, k_1)) \times \\
&\quad \exp(2\pi i b(k_2, x_2)) \exp(-2\pi i b(y_2, k_2)) \quad (6.61)
\end{aligned}$$

Performing the sum over k_1 and k_2 the expression picks up a factor of $\mathcal{D}^2 \delta_{x_1, y_1}$ and $\mathcal{D}^2 \delta_{x_2, y_2}$ according to lemma (6.27). Hence this simplifies:

$$\begin{aligned}
\tau(X_{T_c}) &= (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_l \\
&\quad \exp(2\pi i b(l, x_1)) \exp(2\pi i b(-l, x_2)) \exp(2\pi i q(l)) \times \\
&\quad \mathcal{D}^2 \delta_{x_1, y_1} \mathcal{D}^2 \delta_{x_2, y_2} \quad (6.62)
\end{aligned}$$

Combine the two factors containing b by bilinearity and symmetry:

$$\begin{aligned}
\tau(X_{T_c}) &= (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_l \\
&\quad \exp(2\pi i b(l, x_1 - x_2)) \exp(2\pi i q(l)) \times \mathcal{D}^4 \delta_{x_1, y_1} \delta_{x_2, y_2} \quad (6.63)
\end{aligned}$$

Now rewrite $b(l, x_1 - x_2) = q(l + x_1 - x_2) - q(l) - q(x_1 - x_2)$ to obtain

$$\begin{aligned}
\tau(X_{T_c}) &= (p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_l \\
&\quad \exp(2\pi i q(l + x_1 - x_2)) \exp(-2\pi i q(x_1 - x_2)) \times \mathcal{D}^4 \delta_{x_1, y_1} \delta_{x_2, y_2} \quad (6.64)
\end{aligned}$$

However we have that $\sum_l \exp(2\pi i q(l + x_1 - x_2)) = p_+$ hence we finally obtain (in genus 2)

$$\tau(X_{T_c}) = p_+(p_-)^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \exp(-2\pi i q(x_1 - x_2)) \times \mathcal{D}^4 \delta_{x_1, y_1} \delta_{x_2, y_2} \quad (6.65)$$

The link L has $m = 3$ components and the signature of the linking matrix is $\sigma(L) = 1$. Properly normalized the genus 2 invariant is:

$$\tau(X_{T_c}) = \mathcal{D} \exp(-2\pi i q(x_1 - x_2)) \delta_{x_1, y_1} \delta_{x_2, y_2} \quad (6.66)$$

We have used the fact that $p_+ p_- = \mathcal{D}^2$.

In genus $g \geq 2$ it is necessary to tensor with $g-2$ copies of the genus 1 id computation as in equation (6.24) (however the normalization is not included). This gives a surgery link L with signature $\sigma(L) = 1$ and $m = g+1$ components. The 3-manifold invariant corresponding to a Dehn twist along c_i ($1 \leq i \leq g-1$) is then:

$$\tau(X_{T_{c_i}}) = p_+ p_- \mathcal{D}^{-1-(g+1)-1} \exp(-2\pi i q(x_i - x_{i+1})) \times \mathcal{D}^{2g} \delta_{x_1, y_1} \dots \delta_{x_g, y_g} \quad (6.67)$$

which simplifies to

$$\tau(X_{T_{c_i}}) = \mathcal{D}^{g-1} \exp(-2\pi i q(x_i - x_{i+1})) \times \delta_{x_1, y_1} \dots \delta_{x_g, y_g} \quad (6.68)$$

Lickorish generators and $\mathrm{Sp}(2g, \mathbb{Z})$

In the last subsection the Lickorish generators $\{a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_{g-1}\}$ were studied and their associated projective representations on the Hilbert

space $\mathcal{F}(\Sigma)$ ³ were produced (the matrix elements $\tau(X_{T_{a_i}})$, $\tau(X_{T_{b_i}})$, and $\tau(X_{T_{c_i}})$ were computed explicitly). So we have constructed a map

$$\text{MCG}(\Sigma) \rightarrow \mathbb{PGL}(\mathcal{F}(\Sigma)) \quad (6.69)$$

If Σ is a closed genus g surface then there is a map

$$\text{Sp} : \text{MCG}(\Sigma) \rightarrow \text{Sp}(2g, \mathbb{Z}) \quad (6.70)$$

determined by recording the action of $\text{MCG}(\Sigma)$ only on homology $H_1(\Sigma, \mathbb{Z})$. The kernel of this map is the *Torelli* group, i.e. there is a short exact sequence

$$1 \rightarrow \text{Torelli}(\Sigma) \rightarrow \text{MCG}(\Sigma) \rightarrow \text{Sp}(2g, \mathbb{Z}) \quad (6.71)$$

The map $\text{MCG}(\Sigma) \rightarrow \mathbb{PGL}(\mathcal{F}(\Sigma))$ *factors through* $\text{Sp}(2g, \mathbb{Z})$ if there is a map (broken line) that makes the following diagram commute:

$$\begin{array}{ccc} \text{MCG}(\Sigma) & \xrightarrow{\quad\quad\quad} & \mathbb{PGL}(\mathcal{F}(\Sigma)) \\ & \searrow \text{Sp} & \nearrow \text{---} \\ & \text{Sp}(2g, \mathbb{Z}) & \end{array} \quad (6.72)$$

In genus 1 the Torelli group is trivial. The mapping class group is generated by the s and t matrices:

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (6.73)$$

³see equation (4.90)

which satisfy the relations $(st)^3 = s^2$ and $s^4 = 1$. The Lickorish generators $\{a, b\}$ provide another basis

$$\mathrm{Sp}(T_a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = t \quad \mathrm{Sp}(T_b) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = s^3ts \quad (6.74)$$

In genus g the Torelli group is not usually trivial. However we can still analyze the image of the Lickorish generators in $\mathrm{Sp}(2g, \mathbb{Z})$. The symplectic matrices are as follows:

$$\mathrm{Sp}(T_{a_i}) = \begin{pmatrix} \mathbb{1}_g & \Delta_i \\ 0 & \mathbb{1}_g \end{pmatrix} \quad \text{where } (\Delta_i)_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta = i \\ 0 & \text{otherwise} \end{cases} \quad (6.75)$$

$$\mathrm{Sp}(T_{b_i}) = \begin{pmatrix} \mathbb{1}_g & 0 \\ -\Delta_i & \mathbb{1}_g \end{pmatrix} \quad (6.76)$$

$$\mathrm{Sp}(T_{c_i}) = \begin{pmatrix} \mathbb{1}_g & \Gamma_i \\ 0 & \mathbb{1}_g \end{pmatrix} \quad \text{where } (\Gamma_i)_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta = i \\ 1 & \text{if } \alpha = \beta = i + 1 \\ -1 & \text{if } \alpha = i, \beta = i + 1 \text{ or } \alpha = i + 1, \beta = i \\ 0 & \text{otherwise} \end{cases} \quad (6.77)$$

These matrices can be written in terms of the symplectic basis given in equation (3.75). It is clear that $\mathrm{Sp}(T_{a_i})$ and $\mathrm{Sp}(T_{c_i})$ are already in the symplectic basis by identifying $B = \Delta_i$ and $B = \Gamma_i$, respectively. Denoting the symplectic basis element

$$s_g := \begin{pmatrix} 0 & -\mathbb{1}_g \\ \mathbb{1}_g & 0 \end{pmatrix} \quad (6.78)$$

it is easy to check that $\mathrm{Sp}(T_{b_i}) = s_g^3 \mathrm{Sp}(T_{a_i}) s_g$.

6.3 Main theorem

Theorem 6.79. (Main Theorem) *The group category $\mathcal{C}_{(\mathcal{D},q)}(h, s)$ constructed from the data (\mathcal{D}, q) induces a projective representation of the mapping class group $MCG(\Sigma)$ that is isomorphic to the projective representation of $MCG(\Sigma)$ constructed from toral Chern-Simons theory*

Proof. This is essentially a matter of writing the Lickorish generators (actually their images in the symplectic group, i.e. equations (6.75), (6.76), (6.77)) in terms of the symplectic generators in equation (3.75). We can then use this basis change to compute explicitly what the projective representation (from toral Chern-Simons) found in equations (3.96), (3.97), and (3.98) are in terms of the Lickorish generators.

Once we have this we can compare directly with equations (6.45), (6.55), and (6.68) that were derived from $\mathcal{C}_{(\mathcal{D},q)}(h, s)$.

Manifestly equations (6.75) and (6.77) are already in the form of equation (3.97), and it is straightforward to check that

$$\mathrm{Sp}(T_{b_i}) = s_g^3 \mathrm{Sp}(T_{a_i}) s_g \tag{6.80}$$

Now, using the toral Chern-Simons projective representation in equation (3.97) we see that

$$\widehat{\mathrm{Sp}(T_{a_i})} = \{ \Psi_\gamma(\omega) \rightarrow e^{2\pi i \phi(B)c/24} e^{-2\pi i \Sigma_j B^{jj} q_W(\gamma_j)} e^{-2\pi i \Sigma_{j < k} B^{jk} b(\gamma_j, \gamma_k)} \Psi_\gamma(\omega) \} \tag{6.81}$$

where the hat denotes the operator corresponding to $\text{Sp}(T_{a_i})$. Using $B = \Delta_i$ from above we calculate

$$\widehat{\text{Sp}(T_{a_i})} = \{\Psi_\gamma(\omega) \rightarrow e^{2\pi i\phi(B)c/24} e^{-2\pi i q_W(\gamma_i)} \Psi_\gamma(\omega)\} \quad (6.82)$$

which agrees (up to a projective scalar) with the $\mathcal{C}_{(\mathcal{D},q)}(h, s)$ projective representation in equation (6.45) (notice that the delta functions in equation (6.45) agree with $\gamma \mapsto \gamma$ here).

The toral Chern-Simons matrix in equation (3.97) also implies

$$\widehat{\text{Sp}(T_{c_i})} = \{\Psi_\gamma(\omega) \rightarrow e^{2\pi i\phi(B)c/24} e^{-2\pi i \sum_j B^{jj} q_W(\gamma_j)} e^{-2\pi i \sum_{j < k} B^{jk} b(\gamma_j, \gamma_k)} \Psi_\gamma(\omega)\} \quad (6.83)$$

where we use $B = \Gamma_i$ from above. This becomes

$$\widehat{\text{Sp}(T_{c_i})} = \{\Psi_\gamma(\omega) \rightarrow e^{2\pi i\phi(B)c/24} e^{-2\pi i [q_W(\gamma_i) + q_W(\gamma_{i+1})]} e^{-2\pi i (-1)b(\gamma_i, \gamma_{i+1})} \Psi_\gamma(\omega)\} \quad (6.84)$$

Using the bilinearity of b this is

$$\widehat{\text{Sp}(T_{c_i})} = \{\Psi_\gamma(\omega) \rightarrow e^{2\pi i\phi(B)c/24} e^{-2\pi i [q_W(\gamma_i) + q_W(\gamma_{i+1})]} e^{-2\pi i b(\gamma_i, -\gamma_{i+1})} \Psi_\gamma(\omega)\} \quad (6.85)$$

Using $b(\gamma_i, -\gamma_{i+1}) = q_W(\gamma_i - \gamma_{i+1}) - q_W(\gamma_i) - q_W(-\gamma_{i+1})$ and the fact that for a pure quadratic form $q_W(-\gamma_{i+1}) = q_W(\gamma_{i+1})$ we have

$$\widehat{\text{Sp}(T_{c_i})} = \{\Psi_\gamma(\omega) \rightarrow e^{2\pi i\phi(B)c/24} e^{-2\pi i q_W(\gamma_i - \gamma_{i+1})} \Psi_\gamma(\omega)\} \quad (6.86)$$

which agrees (up to a projective scalar) with the $\mathcal{C}_{(\mathcal{D},q)}(h, s)$ projective representation in equation (6.68).

It remains to compute the toral Chern-Simons matrix $\widehat{\text{Sp}}(T_{b_i})$ using the fact that $\text{Sp}(T_{b_i}) = s_g^3 \text{Sp}(T_{a_i}) s_g$ and equations (3.97) and (3.98). We have

$$\begin{aligned} (\widehat{\text{Sp}}(T_{b_i}))_{\bar{\gamma}} &= (|\mathcal{D}|^{-g/2})^4 \sum_{\gamma', \gamma'', \gamma''' \in \mathcal{D}^g} e^{2\pi i b(\gamma_j, \gamma'_j)} e^{2\pi i b(\gamma'_j, \gamma''_j)} e^{2\pi i b(\gamma''_j, \gamma'''_j)} \times \\ & e^{2\pi i \phi(B)c/24} e^{-2\pi i q_W(\gamma'''_i)} \times e^{2\pi i b(\gamma'''_j, \bar{\gamma}_j)} \end{aligned} \quad (6.87)$$

This is a map *from* a basis of wavefunctions indexed by $\bar{\gamma}$ *to* a basis indexed by γ . The index $j = 1, \dots, g$ counts the factors of \mathcal{D}^g (i.e. $\sum_{\gamma' \in \mathcal{D}^g} = \sum_{\gamma'_1 \in \mathcal{D}} \dots \sum_{\gamma'_g \in \mathcal{D}} = \prod_{j=1}^g \sum_{\gamma'_j \in \mathcal{D}}$). Note that $(|\mathcal{D}|^{-g/2})^4 = \mathcal{D}^{-4g}$. Using lemma (6.27) we can sum over γ' , and then sum over γ'' to obtain

$$\begin{aligned} (\widehat{\text{Sp}}(T_{b_i}))_{\bar{\gamma}} &= \mathcal{D}^{-4g} \mathcal{D}^{2g} e^{2\pi i \phi(B)c/24} \sum_{\gamma''' \in \mathcal{D}^g} e^{2\pi i b(-\gamma_j, \gamma'''_j)} \times \\ & e^{-2\pi i q_W(\gamma'''_i)} \times e^{2\pi i b(\gamma'''_j, \bar{\gamma}_j)} \end{aligned} \quad (6.88)$$

The factor of $\mathcal{D}^{2g} = (\mathcal{D}^2)^g$ appears because the sum over γ' is shorthand for g separate sums $j = 1, \dots, g$. We have

$$(\widehat{\text{Sp}}(T_{b_i}))_{\bar{\gamma}} = \mathcal{D}^{-2g} e^{2\pi i \phi(B)c/24} \sum_{\gamma''' \in \mathcal{D}^g} e^{2\pi i b(\bar{\gamma}_j - \gamma_j, \gamma'''_j)} \times e^{-2\pi i q_W(\gamma'''_i)} \quad (6.89)$$

Likewise the sum over γ''' breaks up as separate sums $j = 1, \dots, g$. We have

$$\begin{aligned}
(\widehat{\text{Sp}(T_{b_i})})_{\bar{\gamma}} &= \mathcal{D}^{-2g} e^{2\pi i \phi(B)c/24} \left(\sum_{\gamma_1''' \in \mathcal{D}} e^{2\pi i b(\bar{\gamma}_1 - \gamma_1, \gamma_1''')} \right) \times \\
&\quad \left(\sum_{\gamma_2''' \in \mathcal{D}} e^{2\pi i b(\bar{\gamma}_2 - \gamma_2, \gamma_2''')} \right) \times \dots \times \left(\sum_{\gamma_{i-1}''' \in \mathcal{D}} e^{2\pi i b(\bar{\gamma}_{i-1} - \gamma_{i-1}, \gamma_{i-1}''')} \right) \times \\
&\quad \left(\sum_{\gamma_i''' \in \mathcal{D}} e^{2\pi i b(\bar{\gamma}_i - \gamma_i, \gamma_i''')} \times e^{-2\pi i q_W(\gamma_i''')} \right) \left(\sum_{\gamma_{i+1}''' \in \mathcal{D}} e^{2\pi i b(\bar{\gamma}_{i+1} - \gamma_{i+1}, \gamma_{i+1}''')} \right) \times \dots \\
&\quad \left(\sum_{\gamma_g''' \in \mathcal{D}} e^{2\pi i b(\bar{\gamma}_g - \gamma_g, \gamma_g''')} \right) \quad (6.90)
\end{aligned}$$

By lemma (6.27) each factor $j \neq i$ is just $\mathcal{D}^2 \delta_{\bar{\gamma}_j, \gamma_j}$. Hence we obtain

$$\begin{aligned}
(\widehat{\text{Sp}(T_{b_i})})_{\bar{\gamma}} &= \mathcal{D}^{-2g} e^{2\pi i \phi(B)c/24} \sum_{\gamma_i''' \in \mathcal{D}} e^{2\pi i b(\bar{\gamma}_i - \gamma_i, \gamma_i''')} \times e^{-2\pi i q_W(\gamma_i''')} \\
&\quad \mathcal{D}^{2(g-1)} \delta_{\bar{\gamma}_1, \gamma_1} \delta_{\bar{\gamma}_2, \gamma_2} \dots \delta_{\bar{\gamma}_{i-1}, \gamma_{i-1}} \delta_{\bar{\gamma}_{i+1}, \gamma_{i+1}} \dots \delta_{\bar{\gamma}_g, \gamma_g} \quad (6.91)
\end{aligned}$$

However $-b(\gamma_i - \bar{\gamma}_i, \gamma_i''') = -q_W(\gamma_i - \bar{\gamma}_i + \gamma_i''') + q_W(\gamma_i''') + q_W(\gamma_i - \bar{\gamma}_i)$ hence substituting we obtain

$$\begin{aligned}
(\widehat{\text{Sp}(T_{b_i})})_{\bar{\gamma}} &= \mathcal{D}^{-2} e^{2\pi i \phi(B)c/24} e^{2\pi i q_W(\gamma_i - \bar{\gamma}_i)} \sum_{\gamma_i''' \in \mathcal{D}} e^{-2\pi i q_W(\gamma_i - \bar{\gamma}_i + \gamma_i''')} \\
&\quad \delta_{\bar{\gamma}_1, \gamma_1} \delta_{\bar{\gamma}_2, \gamma_2} \dots \delta_{\bar{\gamma}_{i-1}, \gamma_{i-1}} \delta_{\bar{\gamma}_{i+1}, \gamma_{i+1}} \dots \delta_{\bar{\gamma}_g, \gamma_g} \quad (6.92)
\end{aligned}$$

The last sum is p_- so we obtain

$$\begin{aligned} (\widehat{\mathrm{Sp}(T_{b_i})})_{\overline{\gamma}} &= p_- \mathcal{D}^{-2} e^{2\pi i \phi(B)c/24} e^{2\pi i q_W(\gamma_i - \overline{\gamma}_i)} \\ &\quad \delta_{\overline{\gamma}_1, \gamma_1} \delta_{\overline{\gamma}_2, \gamma_2} \cdots \delta_{\overline{\gamma}_{i-1}, \gamma_{i-1}} \delta_{\overline{\gamma}_{i+1}, \gamma_{i+1}} \cdots \delta_{\overline{\gamma}_g, \gamma_g} \end{aligned} \quad (6.93)$$

Recall that q_W is pure so $q_W(\gamma_i - \overline{\gamma}_i) = q_W(\overline{\gamma}_i - \gamma_i)$. Hence we obtain

$$\begin{aligned} (\widehat{\mathrm{Sp}(T_{b_i})})_{\overline{\gamma}} &= p_- \mathcal{D}^{-2} e^{2\pi i \phi(B)c/24} e^{2\pi i q_W(\overline{\gamma}_i - \gamma_i)} \\ &\quad \delta_{\overline{\gamma}_1, \gamma_1} \delta_{\overline{\gamma}_2, \gamma_2} \cdots \delta_{\overline{\gamma}_{i-1}, \gamma_{i-1}} \delta_{\overline{\gamma}_{i+1}, \gamma_{i+1}} \cdots \delta_{\overline{\gamma}_g, \gamma_g} \end{aligned} \quad (6.94)$$

which agrees (up to a projective factor) with equation (6.55). \square

Corollary 6.95. *The projective representation of $MCG(\Sigma)$ induced by $\mathcal{C}_{(\mathcal{D}, q)}(h, s)$ factors through the symplectic group, i.e. there is a map CS that makes the following diagram commute:*

$$\begin{array}{ccc} MCG(\Sigma) & \xrightarrow{\mathcal{C}_{(\mathcal{D}, q)}(h, s)} & \mathbb{P}GL(\mathcal{F}(\Sigma)) \\ & \searrow Sp & \nearrow CS \\ & & Sp(2g, \mathbb{Z}) \end{array} \quad (6.96)$$

Alternatively, the Torelli groups acts trivially,

Proof. This is a part of the proof of theorem (6.79) since the toral Chern-Simons projective representation of $Sp(2g, \mathbb{Z})$ provides such a map CS . \square

Appendix A

Remark on Nikulin's Lifting Theorem

The aim here is to slightly revise the main theorem in [BM05] to correct a small error in the statement. The theorem should read

Theorem A.1 (Belov and Moore, 2005). *Classification of quantum toral Chern-Simons:*

1. *The set of ordinary quantum toral Chern-Simons theories is in one-to-one correspondence with trios of data (\mathcal{D}, q, c) where \mathcal{D} is a finite abelian group, q is a **pure** quadratic form, and c is a cube root of the Gauss reciprocity formula.*
2. *The set of spin quantum toral Chern-Simons theories is in one-to-one correspondence with trios of data (\mathcal{D}, q, c) where \mathcal{D} is a finite abelian group, q is a **generalized** quadratic form, and c is a cube root of the Gauss reciprocity formula.*

We have replaced “a quadratic form such that $q(0) = 0$ ” with “a pure quadratic form”. Let us show that this cannot be relaxed.

It is obviously true that if a quadratic form q is pure then $q(0) = 0$. Hence one may wonder if the “pure” condition in theorem (A.1) (see corollary (3.55) for context) can be weakened to “generalized” along with the extra condition that $q(0) = 0$. This is not true. We achieve this by proving a proposition that shows that the conditions in the relevant theorem of Nikulin [Nik80] are sharp.

Before we begin we require the following result (see the appendix in [MH73]):

Theorem A.2 (Milgram). *Let Λ be an **even** lattice, i.e. a lattice equipped with an even symmetric nondegenerate bilinear form $B : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$. Embed Λ in the vector space $V = \Lambda \otimes \mathbb{Q}$. Then by bilinearity B extends to a symmetric nondegenerate bilinear form $B : V \otimes V \rightarrow \mathbb{Q}$. Let $Q : V \rightarrow \mathbb{Q}$ be the induced quadratic refinement defined by $Q(v) := \frac{1}{2}B(v, v)$ for $v \in V$. Let $\text{sign}(B)$ be the signature of (Λ, B) . Consider the discriminant group $\mathcal{D} := \Lambda^*/\Lambda$. B descends to a bilinear form $b : \mathcal{D} \otimes \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$ and Q descends to a **pure** quadratic form $q : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$. It is a fact that the following Gauss formula is satisfied:*

$$\frac{1}{\sqrt{|\mathcal{D}|}} \sum_{x \in \mathcal{D}} \exp(2\pi i q(x)) = \exp(2\pi i \cdot \text{sign}(B)/8) \quad (\text{A.3})$$

Now for the main result of this appendix:

Proposition A.4.

1. *There exists a finite abelian group \mathcal{D} equipped with a generalized quadratic form such that $q(0) = 0$ but q is not pure.*

2. *There exists a finite abelian group \mathcal{D} equipped with a generalized quadratic form such that $q(0) = 0$ but the data (\mathcal{D}, q, C) does not lift to any **even** lattice (where C is determined from q using the Gauss sum formula).*

Proof. Let us begin by proving the first statement. Consider the example $\mathcal{D} = \{0, 1/4, 1/2, 3/4\} = \mathbb{Z}_4$ equipped with the quadratic form

$$q(0) = 0 \pmod{1} \tag{A.5}$$

$$q(1/4) = \frac{7}{8} \pmod{1} \tag{A.6}$$

$$q(1/2) = 0 \pmod{1} \tag{A.7}$$

$$q(3/4) = \frac{3}{8} \pmod{1} \tag{A.8}$$

$$\tag{A.9}$$

A straightforward verification shows that this is a generalized quadratic form, i.e. $q(x+y) - q(x) - q(y) + q(0) = b(x, y)$ is bilinear, and the associated bilinear form on the generator is just

$$b(1/4, 1/4) = \frac{1}{4} \tag{A.10}$$

It is *not pure* (i.e. $q(nx) \neq n^2q(x)$ for every $x \in \mathcal{D}$), but $q(0) = 0$.

Now to show the second claim. Consider the same group and quadratic form (\mathcal{D}, q) . Let us calculate the Gauss sum

$$\frac{1}{\sqrt{|\mathcal{D}|}} \sum_{x \in \mathcal{D}} \exp(2\pi i q(x)) = \exp(2\pi i C/8) \tag{A.11}$$

The LHS is easily computed to equal 1. So we conclude that $C \equiv 0 \pmod{8}$.

Suppose for a contradiction that (\mathcal{D}, q) lifts to an **even** lattice (Λ, B) . By a lift we mean that there is an even lattice (Λ, B) such that the signature of B satisfies

$$\text{sign}(B) \equiv C \equiv 0 \pmod{8} \tag{A.12}$$

and the bilinear form B descends to the bilinear form $b(1/4, 1/4) = \frac{1}{4}$.

On the other hand it is straightforward to compute all possible *pure* quadratic forms on \mathbb{Z}_4 with bilinear form $b(1/4, 1/4) = \frac{1}{4}$ by simply enforcing the purity condition

$$q(nx) = n^2q(x) \tag{A.13}$$

There are two pure quadratic refinements of this b . The first is

$$q_1(0) = 0 \pmod{1} \tag{A.14}$$

$$q_1(1/4) = \frac{1}{8} \pmod{1} \tag{A.15}$$

$$q_1(1/2) = \frac{1}{2} \pmod{1} \tag{A.16}$$

$$q_1(3/4) = \frac{1}{8} \pmod{1} \tag{A.17}$$

$$\tag{A.18}$$

Computing the Gauss sum implies that $C = 1 \pmod 8$. The second is:

$$q_2(0) = 0 \pmod 1 \tag{A.19}$$

$$q_2(1/4) = \frac{5}{8} \pmod 1 \tag{A.20}$$

$$q_2(1/2) = \frac{1}{2} \pmod 1 \tag{A.21}$$

$$q_2(3/4) = \frac{5}{8} \pmod 1 \tag{A.22}$$

$$\tag{A.23}$$

Computing the Gauss sum implies that $C \equiv 5 \pmod 8$.

Now we appeal to theorem (A.2). Consider again the (supposed for contradiction) lift of the original quadratic form q - this is an *even* lattice (Λ, B) with discriminant group $\mathcal{D} = \mathbb{Z}_4$, induced bilinear form $b(1/4, 1/4) = \frac{1}{4}$, and the signature is $\text{sign}(B) \equiv 0 \pmod 8$. Since the lattice is even there is an induced quadratic refinement Q which descends to a *pure* quadratic refinement q . We already calculated all possible pure quadratic refinements for this b (q_1 and q_2 above). Applying the theorem we see that the signature for the lattice must be either

$$\text{sign}(B) \equiv C \equiv 1 \pmod 8 \quad \text{or} \quad 5 \pmod 8 \tag{A.24}$$

which contradicts the fact that we assumed above that the signature must be

$$0 \pmod 8 \tag{A.25}$$

Hence the original quadratic form q does *not* lift to an even lattice. □

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This dissertation was typeset with $\text{\LaTeX} 2_{\epsilon}$ ¹ by the author.

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a trademark of the American Mathematical Society. The macros used in formatting this dissertation were written by Dinesh Das, Department of Computer Sciences, The University of Texas at Austin, and extended by Bert Kay, James A. Bednar, and Ayman El-Khashab.